CONJECTURAL VARIATION MODELS AND SUPERGAMES WITH PRICE-COMPETITION IN A DIFFERENTIATED PRODUCT OLIGOPOLY

Michael Pfaffermayr *)

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Abstract:

Conjectural variation models are popular in empirical research as they infer the degree of market power from real data. IO-theorists, however disapprove it for lack of theoretical foundation arguing that dynamic reactions are forced into a static model with the strategy space and time horizon only loosely defined. The presented model follows an idea put forward by Cabral (1995) and demonstrates that the CV-model can be interpreted as the joint profit maximising steady state reduced form of a price setting supergame in a differentiated product market under optimal punishment strategies. For the symmetric-two firm case the CV-parameter is shown to cover the full range of possible outcomes - from Bertrand competition to joint unconstrained monopoly - depending on the degree of product differentiation, market growth, bankruptcy risk and the discount rate. For the asymmetric cost case numerical calculations are provided.

Keywords: Conjectural variation models, supergames, product differentiation
Jel.: L11, L13, D43, C73

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Austrian Institute of Economic Research, PO.Box 91, A-1103 Vienna, Austria, Email: Pfafferm@wifo.ac.at

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Michael Pfaffermayr

1. Introduction

Conjectural variation models are popular in empirical research as they infer the degree of market power from real data (e.g. Haskel, Martin, 1992 and 1994 or the NEIO-approach of Bresnahan, 1989). IO-theorists, however, disapprove it for lack of theoretical foundation. Especially, it is put forward that dynamics are forced into an essentially static model with the strategy space and the time horizon of the underlying game only loosely defined (if defined at all). This paper wants to build a bridge between conjectural variation models (CV-models) on the one hand and supergames on the other hand. The analysed model follows an idea presented by Cabral (1995) who showed that CV-models can be interpreted as a reduced form of a quantity setting supergame with one-period minimax punishment strategies. Cabral´s model is extended in several important ways to include optimal punishments as well as product differentiation, market growth and bankruptcy risk, and it is shown how the correspondence reads in this richer model.

In particular, this paper demonstrates that the static CV-model may represent the joint profit maximising collusive Nash-equilibrium of a price setting supergame with differentiated products. In contrast, to the seminal papers of Dockner (1992)[1]

1 A similar idea was elaborated by Dockner (1992**).** He formulates a differential game with output-adjustment costs and shows that the subgame perfect equilibrium coincides with the static

and Cabral (1995), however, the supergame analysed here is based on Bertrand-price competition in a differentiated product market and, therefore, allows a wider range of outcomes, from perfect competition to joint, unconstrained monopoly. In addition, it looks at optimal punishment strategies as formulated by Abreu (1988) since the one-period minimax-punishments used in Cabral (1995) may not be optimal. A comprehensive equivalence result is established, which provides a new, sound theoretical basis for analysing the determinants of the crucial, endogenous CV-parameter as a measure of the degree of collusion or equivalently the market power in theoretical models and in empirical studies. The CV-model is introduced in Section 2. Section 3 analyses the corresponding supergame and Section 4 provides a simple example. The last section concludes.

2. Two CV-models

Consider n firms operating in a growing market with product differentiation, facing exogenous risk of bankruptcy and playing an infinite repeated game in prices. The constituent static one-period game is a linear oligopoly (Martin, 1993). At any point in time the inverse demand function of the i-th firm is given by $p_i\left(q_i^t, Q_{-i}^t\right) = a^t - q_i^t - qQ_{-i}^t, \quad i = 1,\ldots,n$. The parameter q, $0 < q \le 1$, defines the degree of product differentiation (taken exogenously) with a low q implying a high degree of product differentiation. Marginal costs, c_i^t, are assumed to be constant but may vary across firms. The corresponding, static CV-model is usually formulated in quantities. Its solution can easily be derived by maximising static per period profits

CV-solution. The main difference to the model presented here is that his model excludes tacit collusion and considers a CV-parameter lower than zero.

of a single firm, i, at any point in time t given the quantities supplied by the remaining firms:

(1) $$\Pi_i^t\left(q_i^t, Q_{-i}^t\right) = \left(a^t - c_i^t - q_i^t - \theta Q_{-i}^t\right) q_i^t$$

The best response of firm i thus amounts to:

(2) $$q_i^{t,CV} = q_i^{CV}\left(Q_{-i}^t; \gamma_i, \theta, a^t, c_i^t\right) = \left(a^t - c_i^t - \theta Q_{-i}^t\right) / \left(2 + \gamma_i \theta\right)$$

with $\gamma_i = \frac{\partial Q_{-i}^t}{\partial q_i^t} = \sum_{j \neq i} \frac{\partial q_j^t}{\partial q_i^t}$ and $i, =1,...,n$.

Proposition 1 below demonstrates that in a differentiated product market for each quantity setting CV-model there exists a corresponding price setting CV-model yielding the same outcome. So it is possible to compare the outcome of the quantity CV-model with that of a supergame based on a price competition in the constituent static one-period game (see also Kamien, Schwartz, 1983 for this relationship).

Proposition 1: Let $\lambda_i = \frac{\partial P_{-i}^t}{\partial p_i^t} = \sum_{j \neq i} \frac{\partial p_j^t}{\partial p_i^t}$ be the price CV-parameter and define $\tilde{\theta} = 1 + \theta(n-2)$. The price setting and the quantity setting CV-models yield identical outcomes, i.e. $p_i^{t,CV'}\left(P_{-i}^t; \lambda_i\right) = p_i^t\left(q_i^{t,CV}\left(Q_{-i}^t\right), Q_{-i}^t; \gamma_i\right)$, if and only if $\lambda_i = \frac{\tilde{\theta}\gamma_i + \theta(n-1)}{1 + \theta\gamma_i}$.

Proof: See the Appendix.

The relationship between λ_i and γ_i resembles the fact that equilibrium prices in Bertrand and Cournot oligopolies do not coincide (Vives, 1985). Setting $\lambda_i = 0$ and $\gamma_i = 0$, θ has to be 0 for proposition 1 to hold. Thus only under complete product differentiation - that means both firms act as monopolists - the equilibrium prices of the two models are identical. For intermediate ranges of product differentiation the price CV-model generally implies some degree of collusion in prices to generate the same equilibrium as the corresponding quantity CV-model. For example, to produce the Cournot-outcome ($g_i = 0$) $l^C = q(n-1) > 0$ has to hold.

3. The price setting supergame with optimal punishments

In the corresponding repeated price setting supergame, firms use a simple stick and carrot pattern as the optimal punishment path. For this to exist (see Abreu 1988, Proposition 2 and Häckner, 1996 for an application to the differentiated product-Bertrand case in a more restricted setting) the following assumptions have to hold.

(A1) There exists a pure strategy equilibrium in the one shot game.

(A2) Each firm's profit function is continuous.

(A3) The strategy set is compact.

Assumption (A1) and (A2) are straightforward in the present setting. The third assumption, however, is more problematic. Collusive prices are bounded from above by the joint profit maximising price. Yet, as negative prices are not admissible (which gives the lower bound) only a limited range of collusive outcomes can be sustained by optimal penal codes[2]. Following Abreu (1988) and Häckner (1996), in constructing optimal punishment strategies it suffices to consider simple stick and carrot strategy profiles. In case of defection of a single firm, there is one-period of punishment and return to collusion the period thereafter. Simultaneous and co-ordinated deviations of a group of players go unpunished, since the employed equilibrium concept is non-cooperative and thus these are irrelevant (Abreu, 1988). The concept of simple strategy profiles means that these are history independent in the sense that they specify the same punishment after any kind of history and any kind of deviation. As shown by Abreu (1988, Proposition 5) the consideration of simple optimal strategy profiles suffices to obtain all perfect equilibrium paths. Due to the requirement of prices to be positive, but lower than or equal to the unconstrained joint profit maximising ones, additional restrictions on the discount rate, the degree of product differentiation, demand and marginal costs have to be introduced. Additionally, punishment prices must be positive. As show below, a sufficient but not necessary condition is that the discount factor, d, is bounded from above and that marginal costs are not too low as compared to a so that punishment prices can be lower than marginal costs without becoming negative.

(A4) $d \leq \bar{d}, c_i^t \geq \frac{a^t}{3}, i = 1,\ldots,n$

Simple, one-period stick and carrot strategies are easily obtained. Suppose there exists an optimal and credible punishment path $s = \{s_1,\ldots,s_n\}$ requiring firm i to follow a price path $p_i^{t,j}, i = 1,\ldots n, j = 1,\ldots n, t = 1,\ldots..$ in case of deviation of firm j.

2 Häckner (1996) constructs more complex strategy profiles to account for punishments, which would imply negative prices in the present setting, by prolonging the number of punishment periods.

Denote further the highest collusive price of firm i that can be sustained by s with $p_i^{t,*}$. In general, this punishment path can be represented by a simpler one, s', consisting of firm i playing $p_i^{1,j,P} < p_i^{1,j}$ in the defection period and $p_i^{t,*}$, $t = 2,...$ in the periods thereafter. Prices and thus profits are equal or higher in the second strategy profile in all subsequent periods after period 1. So for each firm i there exists as a punishment price $p_i^{1,j,P}$ for the first period after defection that equalises the discounted sum of profits of the two strategy profiles and, consequently, makes defection equally costly under both strategy profiles. s' has to form a credible punishment path that is followed by all firms after defection, either in the collusive phase or in the punishment period. To see this note that deviating from the collusion is equally unprofitable under both strategy profiles. On the other hand, with optimal punishments firms should find it advantageous to participate in the price war after defection. Deviating during the one-period price war is unprofitable since deviating under s is unprofitable by assumption and $p_i^{1,j,P} < p_i^{1,j}, i = 1,...,n$, so that deviation under s' it is even less tempting. Therefore, this simple strategy profile defines a credible punishment path. It resembles an important principle stating that optimality requires the deviant to “co-operate“ during the punishment period in his own punishment knowing that the punishment is severe in the first period but in later stages he might have higher profits again. As Abreu (1988) states the principle of “the punishment is made to fit the crime“ is not needed.

Given these arguments the following stick and carrot strategy profile for each firm i is considered:

(i) play the collusive price p_i^t in period t if everyone else, $k \neq i$, has played p_k^t or $p_k^{t,j,P}$ in period *t-1*.

(ii) play $p_i^{t,j,P}$, if firm j has defected either from collusion or from the optimal punishment path in period *t-1* (i.e. start the punishment procedure)

Note that firms are assumed to produce under different marginal costs, so punishment has to be firm-specific and it is indexed accordingly. Before stating the equilibrium conditions, the model-dynamics have to be specified. For this the discount rate is interpreted in broad terms including the growth rate of profits and the risk of bankruptcy (Tirole, 1988). Especially, it is assumed that each firms' demand shifts outward at rate g and as well its marginal costs. So under each regime profits differ by $(1+g)^2$ between two subsequent periods[3], if all firms expand their output and their prices at this constant rate g. Furthermore, firms face exogenous risk of bankruptcy. That means that we can decompose the discount factor into the product of a discount factor proper, $\tilde{d}$, the growth rate of profits and the probability of going out of business. The latter may be defined by an exponential distribution[4] so that the average time of staying in business amounts to $\frac{1}{r}$ for each

3 This follows from the fact that in a linear oligopoly the profit function is quadratic, see equations (5.1) - (5.4) below.

4 Setting $g=0$ (so that $\mathbf{P}_{t.i}=\mathbf{P}_i$ for all *i*) for simplicity and denoting the time in business by *z*, in continuous time, the expected profits in case of collusion amount to

$$E\left[\int_0^\infty e^{-r(t-t)}\mathbf{P}_i\,dt\right]=\int_t^\infty \mathbf{P}_i e^{-r(t-t)}\left[1-P(z\le t)\right]dt=\frac{1}{r}\mathbf{P}_i\int_t^\infty e^{-r(z-t)}\left[1-\left(1-e^{-r(z-t)}\right)\right]dt=\frac{1}{r+\mathbf{r}}\mathbf{P}_i$$

Since $d=\frac{1}{1+r+r}$, the linear approximation around $r=0$ reads $d=\frac{\tilde{d}}{1+\tilde{d}r}\approx\tilde{d}+\frac{\tilde{d}}{\left(1+\tilde{d}r\right)^2}\Big|_{r=0}*r=\tilde{d}(1-r)$.

firm, (Martin, 1996): $d \approx \tilde{d} * (1-r)(1+g)^2$. Formally, the necessary and sufficient conditions for firm i, $i=1,...,n$ playing collusive in the subgame perfect Nash-equilibrium using the simple optimal strategy profile as described above can be stated as (note that prices are measured as deviations from marginal costs and denoted by a bar):

(3) $$\Pi^D_{t,i}\left(\overline{P}^t_{-i}\right) - \Pi_{t,i}\left(\overline{p}^t_i, \overline{P}^t_{-i}\right) \leq d\left[\Pi_{t+1,i}\left(\overline{p}^{t+1}_i, \overline{P}^{t+1}_{-i}\right) - \Pi_{t+1,i}\left(\overline{p}^{t+1,i,P}_i, \overline{P}^{t+1,i,P}_{-i}\right)\right]$$

(4) $$\mathbf{P}^D_{t,i}\left(\overline{P}^{t,j,P}_{-i}\right) - \mathbf{P}_{t,i}\left(\overline{p}^{t,j,P}_i, \overline{P}^{t,j,P}_{-i}\right) \leq d\left[\mathbf{P}_{t+1,i}\left(\overline{p}^{t+1}_i, \overline{P}^{t+1}_{-i}\right) - \mathbf{P}_{t+1,i}\left(\overline{p}^{t+1,j,P}_i, \overline{P}^{t+1,j,P}_{-i}\right)\right]$$

$i = 1,\ldots,n;\ j = 1,\ldots,n;\ t = 1,\ldots$

where the superscript D denotes defection, either from collusion or from the optimal penal code. Profits under defection, punishment, and collusion are defined by (5.1) to (5.4):

(5.1) $$\mathbf{P}^D_{t,i}\left(\overline{P}^t_{-i}\right) = \max_{\overline{p}^t_i} \frac{1}{B}\left(Z^t_i - \tilde{q}\overline{p}^t_i\right)\overline{p}^t_i = \frac{1}{4B\tilde{q}}\left(Z^t_i\right)^2$$

(5.2) $$\mathbf{P}^D_{t+1,i}\left(\overline{P}^{t+1,j,P}_{-i}\right) = \frac{1}{4B\tilde{q}}\left(Z^{t+1,j,P}_i\right)^2 = (1+g)^2\,\mathbf{P}^D_{t,i}\left(\overline{P}^{t,j,P}_{-i}\right)$$

(5.3) $$\mathbf{P}_{t+1,i}\left(\overline{p}^{t+1}_i, \overline{P}^{t+1}_{-i}\right) = \frac{1}{B}\left(Z^{t+1}_i - \tilde{q}\overline{p}^{t+1}_i\right)\overline{p}^{t+1}_i = (1+g)^2\,\mathbf{P}_{t,i}\left(\overline{p}^t_i, \overline{P}^t_{-i}\right)$$

(5.4) $$\mathbf{P}_{t+1,i}\left(\overline{p}^{t+1,j,P}_i, \overline{P}^{t+1,j,P}_{-i}\right) = \frac{1}{B}\left(Z^{t+1,j,P}_i - \tilde{q}\overline{p}^{t+1,j,P}_i\right)\overline{p}^{t+1,j,P}_i = (1+g)^2\,\mathbf{P}_{t,i}\left(\overline{p}^{t,j,P}_i, \overline{P}^{t,j,P}_{-i}\right)$$

using the following abbreviations:

$$Z^t_i = a^t(1-q) - c^t_i\tilde{q} + q\sum_{j=1, j\neq i}^{n} c^t_j + q\overline{P}^t_{-i},\ \ B = \frac{1}{(1+q(n-1))(1-q)},\ \ \tilde{q} = 1 + q(n-2).$$

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using $\tilde{d} = \frac{1}{1+r}$. In this setting the risk of going out of business is exogenous. This is a convenient simplifying assumption. Introducing the firms decisions on how to finance their investments

In (5.1) to (5.4) it is assumed that prices grow at the constant rate *g*, as will be proved to hold in equilibrium. For simplicity the time index is dropped from now on as it is shown that all variables grow at the same growth rate.

The interpretation of the equilibrium conditions is straightforward. Condition (3) simply states that for collusion to be sustainable the additional profit from deviating in a collusive period has to be lower than that lost due to punishment in the next period. Condition (4) establishes the optimal penal code. It says that deviating in the punishment period (triggered by some firm j, $j = 1,....,n$) must yield smaller profit gains for firm i than those lost by starting the punishment phase or the prolonging punishment phase by another period. It should be emphasised that the case $i=j$ is included so that in the perfect equilibrium each firm would find it optimal to participate in its own punishment.

Using a simpler model with one-period minimax, not necessarily optimal, punishments implying zero profits in the punishment phase, Cabral (1995) demonstrates that there exists a "collusive equilibrium" equivalent to a static CV-equilibrium that maximises total industry profits sustainable under collusion. It is defined in a similar way to (3) with the inequality binding. In particular, equivalence is derived by the proposition that there exists a value of the CV-parameter so that CV-best response and the binding non-defection condition of the supergame coincide implying identical equilibria. The CV-model is interpreted as a

(which would be the proper way in models with bankruptcy risk) would complicate the analysis reduced form of a more complex supergame which explicitly places oligopolistic interaction into a dynamic context. Similarly, in the model presented here, the equilibrium conditions lead to an equilibrium relationship in prices equivalent to the reaction curve of a CV-model. We will call it “best collusive response“. Especially, solving (3) and (4) for $\overline{p}_i$ and comparing with (14.6) in the Appendix suggests that also for more general supergames there exists a CV-parameter that reproduces the joint profit maximising Nash-equilibrium path of the supergame. In contrast, to the results of Cabral (1995), however, the derived CV-parameter is not a constant as the best collusive response is non-linear and thus in equilibrium the derived CV-parameter depends on the equilibrium-prices of the supergame.

Proposition 2: Consider a linear oligopoly with product differentiation. Given assumptions (A1) to (A4), there exists a l_i and a g_i such that the static CV-best response is equivalent the best collusive response of a price setting supergame with optimal penal codes that maximises total profits subject to (3) and (4). Especially, there exits a $d \leq \overline{d}$ such that

$$\text{(6.1)}\quad l_i = \frac{2\tilde{q}}{q}\left(\frac{2\sqrt{d\left(Z_i^* - Z_i^N\right)Z_i^N}}{Z_i^* + 2\sqrt{d\left(Z_i^* - Z_i^N\right)Z_i^N}}\right) \Leftrightarrow \overline{p}_i^{CV\prime}\left(Z_i^*; l_i\right) = \overline{p}_i\left(Z_i^*; d\right)$$

$$\text{(6.2)}\quad g_i = \frac{\left(2\sqrt{d\left(Z_i^* - Z_i^N\right)Z_i^N} + Z_i^*\right)q^2(n-1) - 4\tilde{q}\sqrt{d\left(Z_i^* - Z_i^N\right)Z_i^N}}{q\tilde{q}\left(2\sqrt{d\left(Z_i^* - Z_i^N\right)Z_i^N} - Z_i^*\right)} \Leftrightarrow \overline{p}_i^{CV}\left(Z_i^*; g_i\right) = \overline{p}_i\left(Z_i^*; d\right)$$

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considerably.

where $Z_i^* = a(1-q) - c_i \tilde{q} + q \sum_{j=1, j \neq i}^{n} c_j + q\overline{P}_{-i}^*$ results from the joint profit maximising equilibrium of the supergame, Z_i^N is defined analogously for the one shot Nash-equilibrium.

Proof: See the Appendix.

In proving this correspondence it is shown that the best collusive responses given by the solution of (3) and (4) with the inequalities binding give the joint profit maximising prices sustainable under collusion. These can be compared to CV-responses and one can solve easily for the respective CV-parameters.

According to this equivalence result the conjectural variations model has a clear interpretation. The CV-best response represents the joint profit maximising “best collusive response function“ of the supergame and the static equilibrium implied by the CV-model and joint profit maximising equilibrium sustainable under tacit collusion coincide in each period if the CV-parameter is determined as given in proposition 2. In general, however, the one to one correspondence between the “best collusive response“ and the CV-best response depends on the equilibrium strategies and it is not constant across firms. Especially, in the more general supergame it holds not longer true that in the case of cost asymmetry one can derive a unique CV-parameter independent of marginal costs from a given unique discount factor and still maintain an exact relation between the output levels of the CV-model and the supergame as it is the case in Cabral (1995). The reason is that the “collusive best responses“ are non-linear once optimal punishments are introduced instead of minimax-punishments.

The Nash-equilibria of supergames are not unique. Using more sophisticated equilibrium selection approaches may well lead to different equilibria represented by the CV-parameter. For this reason the CV-model derived here has to be interpreted as the upper bound assuming the firms indeed collude to the highest sustainable prices. Without these assumption or without other theories of the equilibrium selection in general it is impossible to draw conclusions on the nature of the supergame from a given CV-parameter.

Proposition 2 also demonstrates that the CV-parameter depends on marginal costs, besides the discount rate, market size and the degree of product differentiation. This casts some doubt on theoretical models assuming heterogeneous firms with different marginal costs but a constant CV-parameter across firms. Similarly, it suggests that empirical models estimating a single CV-parameter from a cross-section of firms or industries or from a pooled cross-section time-series may be misspecified. To analyse the determinants of the degree of collusion as measured by the CV-parameter, it is useful to simplify and restrict the analysis to the case of two firms.

4. A simple example

Assume two firms with equal marginal costs (see also Martin, 1993, pp. 38-40 for a discussion of a similar CV-model). In this case $\tilde{q} = 1$ and

(7) $Z_i = (a-c)(1-q) + b\overline{p}_j = M(1-q) + q\overline{p}_j, i \neq j, i, j = 1,2$

(8) $\overline{p}_i^N = \frac{M(1-q)}{(2-q)}$

(9) $\overline{p}_i^M = \frac{M}{2}$

(10) $\overline{p}_i^* = \overline{p}_i^N \left(1 + \frac{8qd}{(2-q)^2}\right)$

(11) $\overline{p}_i^{i,P} = \overline{p}_i^N \left(1 - \frac{8qd}{(2-q)^2}\right)$

(12) $d \leq \overline{d} = \frac{q}{4-3q}$

Substituting in (10) in (15.1) and (15.2) in the Appendix immediately gives the corresponding CV-values as

(13.1') $l = \frac{(2-q)8d}{(2-q)^2 + 8qd}$

(13.2') $g = -\frac{8d(2+q)(1-q) - q(2-q)^2}{8dq(1-q) - (2-q)^2}$

Figure 1 illustrates the relationship of the static price CV-model and the dynamic supergame for this specific example and demonstrates that in general the CV-parameter is determined by the realised Nash-equilibrium of the supergame. The joint profit maximising prices sustainable under tacit collusion are considered to derive the CV-parameters. By increasing the price CV-parameter the CV-best response curve shifts outward and rotates. Equivalence is attained if it intersects with the non-linear “best collusive responses“. As the Nash-equilibrium is not unique and all price levels between the upper bound and the Nash-prices of the one-shot constituent game can be sustained equilibrium, the derived CV-parameters forms an upper bound as mentioned above.

Insert Figure 1

As can be seen from (13.1), in this simple example the quantity CV-parameter g is determined by the discount rate and the degree of product differentiation. Depending on these determinants it varies between minus one and one and, therefore, in contrast to the model of Cabral (1995) and Dockner (1992) covers the whole range of possible outcomes form pure price-competition, yielding marginal cost prices, to unrestricted joint profit maximising behaviour. From the definition of the discount rate it follows that the market power exerted by firms is higher the higher the discount factor proper, the growth rate of market size[5] (and thus profits) and the lower the risk of bankruptcy. The interpretation is straight forward. A large discount factor proper as well as high growth of profits or low bankruptcy risk imply that future profits get more weight. So defection becomes more costly in terms of forgone profits as compared to that from collusion and a higher degree of collusion is sustainable. Furthermore, in line with other supergames (Häckner, 1996) market power increases with the degree of product differentiation as defection becomes less attractive the more firms are able to differentiate their products. The reason for this is that firms gain additional market shares more easily by cutting their price if their product can easily substituted for that of their rivals. Note further that in the present formulation the CV-parameter is constant over time as in Cabral (1995) as long profits grow at a constant rate and the risk of

5 Following Martin (1993) market size can be defined as the quantity demanded at marginal cost prices: $S = \frac{(a-c)}{(1-q(n-1))}$.

bankruptcy stays constant. Proposition 3 summarises the findings for this simple example.

Proposition 3: Given two symmetric firms and $d < \bar{d}$, the quantity CV-parameter and as well the corresponding price CV-parameter as defined in (13.2), and (13.1) respectively, is higher

(i) the larger the discount factor proper

(ii) the higher the growth of profits (i.e. the faster each firms' market grows)

(iii) the lower the risk of bankruptcy

(iv) the higher degree of product differentiation

The quantity CV-parameter derived form the price setting supergame in differentiated products varies between *-1* and *1*. The corresponding price CV-parameter varies between $\frac{8d}{1+8d}$ and *1*.

Proof: Since $\frac{d\,g}{d\,l} = \frac{1-q}{(1-q\,l)^2} \geq 0$, it suffices to calculate the derivatives of (13.1) which together with the definition of the discount rate proves (i) to (iv). The range of variation of both l and g is easily derived from $q \in [0,1]$ and $d \in [0,\bar{d}]$.•

In asymmetric oligopolies closed solutions to the non-linear conditions for the Nash-equilibrium of the supergame as given by (15.1'') and (15.2'') in the Appendix do not exist. One has to rely on numerical calculations in investigating the effects of the parameter variations on the degree of collusion. Especially, in the asymmetric cost case the CV-parameters depend on the Nash-equilibrium of the underlying supergame, i.e. on all exogenous parameter values, and are firm-specific due to the cost asymmetries. The numerical calculations proceed in three steps: First, the supergame is solved for particular parameter values, varying marginal costs, the degree of product differentiation and the discount rate. Secondly, the corresponding CV-parameters are derived. In the third step it is investigated whether the widely used CV-model with constant CV-parameter across firms (derived from the symmetric case) gives a fair approximation to the optimal equilibrium of the repeated game.

Tables 1 and 2 exhibit the calculations for parameter values in an intermediate range where the conditions of Proposition 2 hold. The numerical calculations indicate that for the chosen parameter values the comparative static results of Proposition 3 carry over to the asymmetric cost case. Increasing the discount factor (or equivalently the discount factor proper, market growth or bankruptcy risk) increases the price CV-parameter of both the efficient and the less efficient firm (Table 1). The same holds true for increasing product differentiation (Table 2). Compared to the symmetric case the more efficient firm now exhibits a lower CV-parameter and charges a lower price as it can gain more from defection and

Table 1: The CV-model and the supergame: 2 firms with different costs, variations in *d*

	Supergame				CV-Model with symmetric price CV-parameter	
	Price CV-parameter: $(l_{1,} l_2)$		Prices: (p_1, p_2)		Prices: (p_1, p_2)	
c_2	*d*=0.1	*d*=0.2	*d*=0.1	*d*=0.2	*l*=0.46	*l*=0.72
0.5	0.46, 0.46	0.72, 0.72	0.64, 0.64	0.68, 0.68	0.64, 0.64	0.68, 0.68
0.6	0.40, 0.53	0.66, 0.78	0.66, 0.69	0.69, 0.72	0.66, 0.69	0.70, 0.72
0.7	0.31, 0.63	0.53, 0.82	0.67, 0.73	0.70, 0.75	0.68, 0.74	0.72, 0.76

Note: $a = 1, c_1 = 0.5, q = 0.75$. The non-linear system of equations defined by (15.1'')-(15.2'') in the Appendix is solved numerically using the Gauss-Newton method with the unconstrained joint profit maximising prices as starting value.

faces less severe punishments for defection from the other, less efficient firm. The reverse holds true for the less efficient firm, it behaves more co-operatively. Note that this pattern can be found for all parameter combinations and it is preserved for variations in both the discount rate and in the product differentiation parameter.

For a low degree of product differentiation ($q = 0.8$) Table 2 additionally shows that there exists a hump shaped relationship between the CV-parameter and the degree of cost asymmetry for some parameter combinations. With a low degree of product differentiation the CV-parameter of the less efficient firm first rises and then falls with c.p. increasing own marginal costs. In the presence of large cost differences the efficient firm is less willing to play co-operatively, especially if products are not differentiated much. Its low costs and thus low equilibrium prices

Table 2: The CV-model and the supergame: 2 firms with different costs, variations in q

	Supergame				CV-Model with symmetric price CV-parameter	
	Price CV-parameter: (l_1, l_2)		Prices: (p_1, p_2)		Prices: (p_1, p_2)	
c_2	q=0.75	q=0.80	q=0.75	q=0.80	l=0.72	l=0.71
0.5	0.72, 0.72	0.71, 0.71	0.68, 0.68	0.66, 0.66	0.68, 0.68	0.66, 0.66
0.6	0.66, 0.78	0.64, 0.76	0.69, 0.72	0.67, 0.70	0.70, 0.72	0.68, 0.70
0.7	0.53, 0.82	0.43, 0.59	0.70, 0.75	0.67, 0.72	0.72, 0.76	0.70, 0.75

Note: $a = 1, c_1 = 0.5, d = 0.2$. The non-linear system of equations defined by (15.1'')-(15.2'') in the Appendix is solved numerically using the Gauss-Newton method with the unconstrained joint profit maximising prices as starting value.

then more than outweigh the influence of the higher marginal costs of the less efficient firm in determining its CV-parameter. Note however, that this does not hold true for the equilibrium prices of the low cost – firm. Here the impact of the high own marginal costs dominates and the price charged by the less efficient firm increases in its marginal costs.

The last two columns in Table 1 and 2 refer to the approximation of the supergame by a CV-model with a symmetric CV-parameter but different marginal costs - the model widely used in empirical IO-research. The CV-parameter is derived form a supergame with equal firms (first line in the Tables) and with this CV-parameter the static price CV- model is solved for the asymmetric cost case. For the chosen range of parameter values this CV-model gives a reasonable approximation to the optimal equilibrium of the supergame, especially if the cost asymmetry is not too large. Generally, the equilibrium prices, in particular that of the more efficient firm, seem to be slightly overestimated.

5. Conclusions

This paper fortifies the bridge between CV-models and supergames. It interprets the static CV-model as a reduced form of a price setting supergame in a differentiated product market and in this way provides a comprehensive theoretical foundation of the widely criticised static CV-models. More important, however, this approach allows to analyse the economic determinants of the differences in profitability by deriving the determinants of the crucial CV-parameter from those factors determining the outcome of the underlying supergame. This gives a new theoretical basis for empirical research in profitability differences previously based on the structure-conduct-performance approach and provides an enhanced menu of testable hypotheses.

This study uses a differentiated Bertrand oligopoly as constituent game of a supergame and considers optimal punishment strategies to set up the corresponding price and quantity CV-model. It is shown that depending on the degree of product differentiation, the discount factor proper, market growth and bankruptcy risk the whole range of possible outcomes, from marginal cost prices to unconstrained joint profit maximising ones, can be explained by the supergame and mapped to the corresponding static CV-model. The correspondence is derived by defining the “collusive best response“ of the supergame as the equilibrium relationship with the non-defection conditions binding. In general it is non-linear once optimal punishments are introduced implying that the corresponding CV-parameters depend on the considered Nash-equilibrium of the supergame and, therefore, are firm-specific. This casts some doubt on both theoretical and empirical models which assume heterogeneous firms but a constant CV-parameter across firms. The numerical calculations, however, show that the simple CV-model with symmetric conjectural variations, but different marginal costs seems to provide reasonable approximations, especially if cost asymmetry is not too large. In future research this numerical simulations should be extended to cover a wider range of parameters and evaluate the econometric performance of the CV-model under the assumption that data are generated by the supergame discussed above.

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Appendix

Proof of Proposition 2:

To simplify notation the time index is suppressed. The inverse of the demand function of firm i is given by $q_i = \frac{a(1-q) - p_i\tilde{q} + qP_{-i}}{(1+q(n-1))(1-q)}$. Aggregating over i leads to

(14.1) $Q = \frac{an-P}{1+q(n-1)}$ with $P = \sum_{i=1}^{n} p_i$, $Q = \sum_{i=1}^{n} q_i$.

Inserting (14.1) in (2) and defining $Z_i = a(1-q) - c_i\tilde{q} + qc_{-i} + q(P_{-i} - c_{-i})$ with $P_{-i} = \sum_{i \neq j, j=1}^{n} p_j$ and $c_{-i} = \sum_{i \neq j, j=1}^{n} c_j$ (2) can be reformulated as:

(14.2) $q_i^{CV} = \frac{Z_i + q\left(p_i^{CV} - c_i\right)}{(2-q(1-g_i))(1+q(n-1))}$.

On the other hand, demand of firm i is given by

(14.3) $q_i^{CV} = \frac{Z_i - \tilde{q}\left(p_i^{CV} - c_i\right)}{(1-q)(1+q(n-1))}$.

Equating (14.2) and (14.3) and solving for $\overline{p}_i^{CV} = p_i^{CV} - c_i$ leads to the best response of the quantity CV-model in terms of prices (Note that from now on all prices are expressed as deviations from marginal costs):

(14.4) $\overline{p}_i^{CV}\left(Z_i ; g_i\right) = \frac{Z_i(1+qg_i)}{\tilde{q}(2+qg_i) - q^2(n-1)}$.

The price setting CV-model implies the best responses:

(14.5) $\overline{p}_i^{CV'}\left(Z_i ; l_i\right) = \frac{Z_i}{2\tilde{q} - ql_i}$

Equating (14.4) and (14.5) shows that the quantity stetting CV-model and the price setting CV-model generate the same outcome, if and only if

(14.6) $l_i = \frac{\tilde{q}g_i + q(n-1)}{1+qg_i}$.•

Proof of Proposition 2:

The proof proceeds in several steps. To keep notation simple the time index is skipped.

(i) Solving (3 and (4) for $j=i$ under the assumption that both hold with equality yields:

(15.1) $\bar{p}_i = \dfrac{Z_i + \sqrt{d\left(Z_i^2 - Z_i^{P,2}\right)}}{2\tilde{q}}$

(15.2) $\bar{p}_i^{i,P} = \dfrac{Z_i^P - \sqrt{d\left(Z_i^2 - Z_i^{P,2}\right)}}{2\tilde{q}}$

where the larger and the smaller root, respectively, is taken according to optimality. Adding the best response of the one-shot game, $2\tilde{q}\bar{p}_i^N = Z_i^N$, to (15.1) and (15.2) gives:

(15.1') $\left(\bar{p}_i - \bar{p}_i^N\right)2\tilde{q} = Z_i - Z_i^N + \sqrt{d\left(Z_i^2 - Z_i^{P,2}\right)}$

(15.2') $\left(\bar{p}_i^N - \bar{p}_i^{i,P}\right)2\tilde{q} = Z_i^N - Z_i^P + \sqrt{d\left(Z_i^2 - Z_i^{P,2}\right)}$

Subtracting (15.2') from (15.1') and aggregating over i shows that for all i it holds that

(16.1) $\bar{p}_i^{i,P} = 2\bar{p}_i^N - \bar{p}_i$

and therefore

(16.2) $Z_i^P = 2Z_i^N - Z_i$

Inserting (16.2) in (15.1) and (15.2) then leads to

(15.1'') $\bar{p}_i = \dfrac{Z_i + 2\sqrt{d\left(Z_i - Z_i^N\right)Z_i^N}}{2\tilde{q}}$

(15.2'') $\bar{p}_i^{i,P} = \dfrac{Z_i^P - 2\sqrt{d\left(Z_i^N - Z_i^P\right)Z_i^N}}{2\tilde{q}}$

(ii) For $d = 0$ (15.1'') implies the Nash-equilibrium of the one-shot game as the supergame-equilibrium. By continuity of (15.1'') there exists a d small enough such that equilibrium prices are smaller than the unconstrained joint profit maximising ones. For such a d (3) and (4) hold with equality if firms maximise joint profits subject to these constraints. To see this assume there is a slack in (3) but not (4) for firm i and that all other constraints are binding. Inserting (3) in (4) results in

(17) $F\left(\bar{p}_i, \bar{P}_{-i}\right) = \mathbf{P}_i^D\left(\bar{P}_{-i}\right) + \frac{d}{1-d}\mathbf{P}_i^D\left(\bar{P}_{-i}^{i,P}\right) - \frac{1}{1-d}\mathbf{P}_i\left(\bar{p}_i, \bar{P}_{-i}\right) \le 0$

Differentiation of (17) by the price of firm *i* and firm $j \neq i$ yields

$$(18.1)\ \frac{\P F}{\P \, \overline{p}_i} = \frac{1}{\tilde{B}(1-d)}\left(2\tilde{q}\overline{p}_i - Z_i\right) = \frac{2\tilde{q}}{\tilde{B}(1-d)}\left(\overline{p}_i - \overline{p}_i^{D}\right) \geq 0$$

$$(18.2)\ \frac{\P F}{\P \, \overline{p}_j} = \frac{q}{\tilde{B}2\tilde{q}}\left(Z_i - \frac{2\tilde{q}\overline{p}_i}{(1-d)}\right) = \frac{q}{\tilde{B}2\tilde{q}}\left(2\tilde{q}\overline{p}_i^{D} - \frac{2\tilde{q}\overline{p}_i}{(1-d)}\right) = \frac{q}{\tilde{B}(1-d)}\left(\overline{p}_i^{D}(1-d) - \overline{p}_i\right) \leq 0 \,.$$

So firm *i* could increase its price and as well the other firms without violating their binding constraints. This increases total profits if $d\mathbf{P} = \frac{1}{B}\sum_{i=1}^{n}\left[Z_i - 2\tilde{q}\overline{p}_i + \sum_{k=1,k\neq i}^{n} q\overline{p}_k\right]d\overline{p}_i \geq 0$. To show this assume all firms increase $\overline{p}_i$ by the same small amount, i.e. $d\overline{p}_i = d\overline{p}$. In this case we have

$$\frac{d\mathbf{P}}{d\overline{p}} = \frac{1}{B}\sum_{i=1}^{n}\left[Z_i - 2\tilde{q}\overline{p}_i + \sum_{k=1,k\neq i}^{n} q\overline{p}_k\right] = \sum_{i=1}^{n} a(1-q) - c_i\tilde{q} + q\sum_{j=1,\,j\neq i}^{n} c_j + 2q\overline{P}_{-i} - 2\tilde{q}\overline{p}_i =$$
$$\sum_{i=1}^{n} a(1-q) - c_i\left(1+q(n-1)\right) + qC + 2q\overline{P} - 2\left(1+q(n-1)\right)\overline{p}_i =$$
$$(1-q)\left(an - C - 2\overline{P}\right) \geq (1-q)\left(an - C - 2\overline{P}^{M}\right) = 0$$

using the abbreviations $C = \sum_{j=1}^{n} c_j$, $\overline{P} = \sum_{i=1}^{n} \overline{p}_i$ and the superscript *M* to denote unconstraint joint profit maximising prices.

On the other hand assume a slack in (5) but not in (4) for firm *i*. Define

$$(5')\quad G = \mathbf{P}_i^{D}\left(\overline{P}_{-i}^{i,P}\right) - (1-d)\mathbf{P}_i\left(\overline{p}_i^{i,P}, \overline{P}_{-i}^{i,P}\right) - d\mathbf{P}_i\left(\overline{p}_i, \overline{P}_{-i}\right) \leq 0$$

Differentiation of (5') and (17) results in

$$(19.1)\ \frac{\P G}{\P \, \overline{p}_i^{i,P}} = \frac{(1-d)}{\tilde{B}}\left(2\tilde{q}\overline{p}_i^{i,P} - Z_i^{P}\right) \leq -\frac{(1-d)}{\tilde{B}}\sqrt{d\left(Z_i^2 - Z_i^{P,2}\right)} \leq 0$$

$$(19.2)\ \frac{\P G}{\P \, \overline{p}_j^{i,P}} = \frac{q}{\tilde{B}2\tilde{q}}Z_i^{P} - \frac{q(1-d)\overline{p}_i^{i,P}}{\tilde{B}} = \frac{(1-d)}{\tilde{B}}\sqrt{d\left(Z_i^2 - Z_i^{P,2}\right)} + \frac{qd\overline{p}_i^{i,P}}{\tilde{B}} \geq 0$$

$$(19.3)\ \frac{\P F}{\P \, \overline{p}_j^{i,P}} = \frac{dZ_i^{P}}{\tilde{B}2\tilde{q}(1-d)} \geq 0$$

Therefore, all firms could lower their punishment prices, which produces a slack in (3) and according to (19.3) produces also one in (17) so that the same argument as above applies.

Therefore, it can be concluded that in the sustainable, joint profit maximising equilibrium the restrictions (3) and (4) are binding.

(iii) The punishment price is restricted to be positive. In addition (15.1'') is bounded from above by the unconstrained joint profit maximising price, $\overline{p}_i^M$. The latter is given by (using $M_i = a - c_i$): $\overline{p}_i^M = \frac{M_i}{2}$. Define $X_i = q\overline{P}_{-i}^M = \frac{q}{2} \sum_{j\neq i, j=1}^{n} M_j$ and $Z_i^o = a(1-q) - c_i \tilde{q} + q \sum_{j=1, j\neq i}^{n} c_j$ so that $Z_i^M = Z_i^o + X_i$. There exists a $\overline{d} \leq 1$ such that $\overline{p}_i \leq \frac{M_i}{2}$ and $\overline{p}_i^{i,P} \geq -c_i$, if $a \leq 3c_i, i = 1,\ldots,n$. $\overline{d}$ is sufficient however not necessary for the joint profit maximising cartel to be sustainable under the restrictions (4) and (5). It can be derived as follows:

(20.1) $2\tilde{q}\overline{p}_i = Z_i + \sqrt{d\left(Z_i^2 - Z_i^{P,2}\right)} \leq Z_i^M + \sqrt{d\left[\left(Z_i^M\right)^2 - \left(Z_i^0\right)^2\right]}$

Since $Z_i^M - 2\tilde{q}\overline{p}_i^M = M_i(1-q) + qnM_i - q\sum_{j=1}^{n} M_j + X_i - \left(1 + q(n-2)\right)M_i =$

$qM_i - q\sum_{j=1}^{n} M_j + \frac{q}{2}\sum_{i\neq j, j=1}^{n} M_j = -X_i$. So $\overline{p}_i \leq \overline{p}_i^M$ if $Z_i^M - 2\tilde{q}\overline{p}_i^M + \sqrt{d\left[\left(Z_i^M\right)^2 - \left(Z_i^0\right)^2\right]} =$

$-X_i + \sqrt{d\left[\left(Z_i^0 + X_i\right)X_i\right]} \leq 0$ which is the case if $d \leq \overline{d}_i = \frac{X_i}{2Z_i^0 + X_i}$.

From (16.1) we have

(20.2) $\overline{p}_i^{i,P} = 2\overline{p}_i^N - \overline{p}_i \geq -\overline{p}_i^M = -\frac{1}{2}(a - c_i) \geq -c_i \Leftrightarrow a \leq 3c_i$.

Therefore, if $d \leq \overline{d} = \min_i\left[\overline{d}_i\right]$ and $a \leq 3c_i$ for all i then $p_i \leq p_i^M$ and $p_i^{i,P} \geq 0$ for $i = 1,\ldots,n$.

(iv) The comparison of (15.1'') with the static best response of the price CV-model as given in (14.6) indicates that the static best response of the CV-model and the "collusive best response" of the supergame coincide and thus imply the same equilibrium prices if the price CV-parameter fulfils (6.1). By proposition 1 the quantity CV-parameter is then given by (6.2). On the other hand assume the solutions of the CV-model and the supergame coincide. Since each equilibrium price vector uniquely identifies the corresponding CV-parameters (6.1) and (6.2) are necessary *and* sufficient for equivalence.

(v) It remains to show that equilibrium prices grow at rate g. Assume that a, c_i grow at rate g and that all firms increase the prices (both in case of collusion and defection) at this rate from period t to $t+1$ with exception of firm i. Inserting (4) in (5) for period $t+1$ (assuming no risk of going out of business for simplicity) the equilibrium condition in period t (17) reads:

$$(17') \quad \left[\mathbf{P}^{D}_{t,i}\left(P^{t}_{-i}\right) + \frac{\tilde{d}}{1-\tilde{d}} \mathbf{P}^{D}_{t,i}\left(P^{t}_{-i}\right) - \frac{1}{1-\tilde{d}} \mathbf{P}_{t,i}\left(\frac{p^{t}_{i}}{1+g}, P^{t}_{-i}\right)\right](1+g)^{2} \leq 0$$

Since the first derivative of (17') is positive at $p^{t}_{i} \leq p^{t+1}_{i}$ as shown in (18.1), the same argument as in (ii) applies. Especially, firm i could increase its price at rate g without violating the constraints and joint profit maximisation requires firm i to do so.

(vi) If there is a risk of bankruptcy, firms discount expected future profits accordingly as given in footnote 2.•

Figure 1: CV-model and supergame with two symmetric firms