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The paper offers a complete analysis of the welfare-maximising capital investment and resource depletion policies in the Dasgupta-Heal-Solow-Stiglitz (DHSS) model with capital depreciation and any returns to scale. We establish a general existence result and show that an optimal admissible policy may not exist if the output elasticity of the resource equals 1. We characterise the optimal policies by applying an appropriate version of the Pontryagin maximum principle for infinite-horizon optimal control problems. We conclude the paper with an economic interpretation and a discussion of the welfare-maximising policies.

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The Optimal Use of Exhaustible Resources under Nonconstant Returns to Scale

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October 3, 2016

Abstract

The paper offers a complete analysis of the welfare-maximizing capital investment and resource depletion policies in the Dasgupta–Heal–Solow–Stiglitz (DHSS) model with capital depreciation and any returns to scale. We establish a general existence result and show that an optimal admissible policy may not exist if the output elasticity of the resource equals one. We characterize the optimal policies by applying an appropriate version of the Pontryagin maximum principle for infinite-horizon optimal control problems. We finish the paper with an economic interpretation and a discussion of the welfare-maximizing policies.

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1 Introduction

The 1972 Club of Rome's report on the 'Limits to Growth' predicted a gloomy future for the world and its resources, sparking an ongoing controversial debate (Meadows et al. [39]). Essential aspects of the rise and decline scenario were increasing resource scarcity and pollution. The central question was whether the scarcity of natural resources such as fossil fuels would limit growth and cause a substantial decline in standard of living. The report has been subject to the utmost scrutiny by academics and journalists (Bardi [16]). Recent re-examinations by Hall and Day [29] and Turner [50] lend credibility to some of the conclusions reached in the report.

Following the publication of the report, a group of distinguished economists countered the scarcity argument in a series of papers now commonly referred to as the Dasgupta–Heal–Solow–Stiglitz (DHSS) model (see, [24,47,49]). Their efforts resulted in a substantial theoretical contribution to the theory of economic growth and a standard model with resource constraints. One argument countering the dominant pessimism was that man-made capital could substitute resources in the production of consumption goods. This led to the concept of a weakly sustainable path, based on which the expansion of man-made capital can offset the depletion of resources (for a critique, see Beckerman [18] and Asheim et al. [14]). A weakly sustainable path may not be economically optimal or welfare-maximizing. Finding welfare-maximizing policies requires solving the model as an infinite-horizon optimal control problem. The assumption of an infinite horizon is natural for processes that continue for an indefinite, but not necessarily infinite, period of time. The optimal investment policy tells us what portion of final output should be invested in the produced capital. The optimal depletion policy specifies the rate of extraction of a finite stock of exhaustible resources. Both policies are time-consistent under the standard exponential discounting.

Despite an extensive theoretical investigation, a complete analysis of the model was only recently presented by Benchekroun and Withagen [19] for constant returns to scale and in the absence of capital depreciation. Constant returns imply a concave Hamiltonian, which, together with appropriate transversality conditions, allows the application of the standard *sufficient* optimality conditions known as Arrow's sufficiency theorem (Seierstad and Sydsæter [45]). Decreasing or increasing returns to scale typically yield a nonconcave Hamiltonian, thereby precluding the application of Arrow's theorem. It is clear that the standard approach must be discarded in favor of an *existence* theorem followed by the application of *necessary* optimality conditions. The caveat is that no ready-to-use existence theorem is available.

The lack of an existence result is not surprising given that the DHSS model contains many features that are completely nonstandard in optimal control theory. The integral constraint on the extraction rate as a control variable is not covered by standard existence theorems; for example, Balder's theorem [15] cannot be directly applied in this situation. Moreover, the absence of pointwise bounds on the control variable leads to a violation of the uniform integrability conditions in Balder's theorem in the case of unit output elasticity of the resource, as well as in the general case, when we reduce the problem with an integrally constrained control variable to a problem with a locally integrable control variable.

To the above problem we may add the singularity in the logarithmic instantaneous utility function. Note that the logarithmic utility function is the simplest case of a utility function having constant elasticity of intertemporal substitution. Maximizing aggregate discounted logarithmic utility over a time interval is equivalent to maximizing aggregate discounted future growth rates. This is consistent with an economy trying to grow as quickly as possible, a plausible objective given the primacy of growth as an indicator of economic development.

Furthermore, the general variant of the maximum principle in Halkin [28] does not guarantee normality of the optimization problem and the standard transversality conditions at infinity, which makes the application of the maximum principle ineffective. Problems arising in the application of the theory to models of economic growth with resource constraints are discussed in Section 2.

The centerpiece of the paper is a complete and rigorous study of the DHSS model under decreasing or increasing returns to scale. The maximum returns to scale that we consider are equal to 2, which is sufficient given that the existing empirical evidence typically shows slightly increasing returns on an aggregate level. For example, Antweiler and Trefler [2] use international trade flows to estimate the aggregate returns to scale at 1.05. Their model includes eleven production factors, ranging from capital stock, four categories of human capital, three energy stocks (coal reserves, oil and gas reserves and hydroelectric potential) and three types of land (cropland, pasture and forests). Laitner and Stolyarov [37] use stock market data to estimate 1.09–1.11 as a range for the aggregate returns to scale. Their study uses labor and a composite stock of tangible and intangible assets as inputs. The question of returns to scale is not entirely settled—it will depend on the country, the industry and sometimes even the firm. For example, Zhang et al. [51] report increasing returns to scale for each of the seventeen sectors of the Chinese economy. Whereas economy-wide studies often report increasing returns, firm-level ones may find constant or slightly decreasing returns (Basu and Fernald [17]). Firm-level studies typically include labor, capital and materials as production factors related to natural stock.

Briefly anticipating the results, let us state that the optimal solutions to the DHSS model do not qualitatively depend on returns, unless one of the two output elasticities equals one. The case of a unit output elasticity of capital corresponds to the case of strong scale effects discussed in Jones [35]. Strong scale effects cannot be discarded on theoretical grounds due to the compound nature of produced capital in the model, which comprises physical capital, human capital and a stock of knowledge. Strong scale effects in the accumulation of knowledge arise due to a nonrival nature of ideas (Jones [35]). Weak scale effects, on the other hand, imply that the expansion of knowledge becomes successively more difficult to achieve due to a 'fishing out' of ideas (Jones [34]; Fernald and Jones [25]). The version of the DHSS model studied by Benchekroun and Withagen [19] under constant returns to scale does not include capital depreciation. Although it can be shown that a constant depreciation rate does not change the nature of solutions under constant returns to scale, it will substantially alter the dynamics of the model and its asymptotic properties under nonconstant returns. The case of a unit output elasticity of the resource is included because in this scenario an optimal solution does not exist for a sufficiently small initial stock of produced capital. This implies, curiously enough, that it is impossible to formulate a welfare-maximizing policy at an early stage of economic development when produced capital is scarce and resources are abundant. In the model, this essentially implies an initial jump to the minimal stock of produced capital, followed by an optimal policy thereafter.

For optimal growth models with an exhaustible resource, a general approach to the analysis based on reducing the model to a single dimension, establishing the existence of a solution and applying necessary optimality conditions has been pioneered in Aseev, Besov and Kaniovski [7]. Their model stands in the tradition of the growth model by Grossman and Helpman [26], which includes a fixed supply of labor that can be used either in the production of the final output or in the generation of knowledge. The exhaustible resource enters the model as a production factor, whose input is subject to a constraint imposed by a finite stock. In the most realistic case, when technical progress either depends on the resource or shows weak scale effects, the output along the optimal path first rises and then declines. This rise and decline scenario along the optimal path has already been noted in the original analysis of the DHSS model.

The paper is organized as follows. After a methodological discussion in Section 2 and the statement of the DHSS model in Section 3, in Section 4 we rewrite the model in terms of a single state variable representing a ratio of stocks in appropriate powers. In Section 5 we present a direct solution of the pure resource depletion problem. Then Section 6 investigates the existence of an optimal admissible control in the general case. The existence result can be derived from Balder's theorem [15], but we use a theorem by Besov [20] with slightly easier-to-verify conditions. We then use a variant of the Pontryagin maximum principle for infinite-horizon problems by Aseev and Veliov [13] and study the Hamiltonian system in Sections 7 and 8. It turns out that, in conjunction with the discount and depreciation rates, different degrees of substitutability between the factor inputs lead to qualitatively distinct dynamics of the above-mentioned stock ratio under welfare-maximizing policies, which either tends to a finite value or goes to infinity. The final section (Section 9) offers a sensitivity analysis (comparative statics) of the welfare-maximizing policies.

2 Methodological pitfalls

The use of optimal control theory to obtain welfare-maximizing solutions to problems of economic growth has a long tradition in economics. This line of research was initiated by Ramsey [43] and continued by Cass [22], Koopmans [36], Shell [46], Arrow and Kurz [4], and has become the standard method of solving optimal economic growth models.

As a rule, models of optimal economic growth are formulated as optimal control problems with *infinite time horizons*. The early literature has recognized pitfalls in the application of optimal control theory to such problems. The maximum principle for infinite-horizon problems may not hold in the normal form, and the standard transversality conditions at infinity may fail (see Halkin [28], Michel [40] and Shell [46]). Additional difficulties arise when the model involves an exhaustible resource as an essential factor of production. In this case, typically, admissible controls are only bounded in an integral sense, which precludes the direct application of the standard existence results.

In this section we introduce some basic concepts and recent results of infinite-horizon optimal control theory that will be used throughout the paper. We also provide two examples showing the difficulties akin to infinite-horizon optimal economic growth models with exhaustible resources. The first example illustrates possible inconsistencies between the standard transversality conditions and the core conditions of the maximum principle in the case of infinite-horizon optimal growth models with resource constraints. The second example shows that an optimal admissible control may not exist when admissible controls are only bounded in the integral sense, whereas a formal application of the general maximum principle can yield misleading results in this case.

2.1 A general model of optimal growth

Take the following general model of optimal growth as an infinite-horizon optimal control problem (P):

$$J(x(\cdot),u(\cdot)) = \int_0^\infty e^{-\rho t} g(x(t),u(t))\,dt \to \max, \tag{2.1}$$

$$\dot{x}(t) = f(x(t),u(t)), \qquad x(0) = x_0, \tag{2.2}$$

$$u(t) \in U, \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the values of a state variable and a policy variable at time $t \geq 0$, x_0 is a given initial state, U is a given nonempty subset of $\mathbb{R}^m$, and $\rho > 0$ is a discount rate. The coordinates of the vector $x(t)$ represent the amounts of different factor inputs used to produce a final consumption good. The inputs may include stocks of physical capital, human capital, knowledge or natural resources. The coordinates of the policy vector $u(t)$ give instantaneous changes in stocks, such as an investment rate or a depletion rate.

Let G be a given nonempty open convex subset of $\mathbb{R}^n$ of possible values of the state variable $x(\cdot)$. Assume that the functions $f\colon G \times U \to R^n$ and $g\colon G \times U \to \mathbb{R}^1$ and their partial derivatives $f_x(\cdot,\cdot)$ and $g_x(\cdot,\cdot)$ are continuous on $G \times U$, and that $x_0 \in G$. The vector function $f(\cdot,\cdot)$ describes the dynamics of different production factors, whereas the scalar function $g(\cdot,\cdot)$ gives the instantaneous utility incurred by $x(t)$ and $u(t)$.

Any Lebesgue measurable function $u\colon [0,\infty) \to \mathbb{R}^m$ satisfying condition (2.3) for all $t \geq 0$ will be called a *control*. If $u(\cdot)$ is a control, then the corresponding *trajectory* is a locally absolutely continuous solution $x(\cdot)$ of the initial value problem (2.2), which is defined on some finite or infinite time interval $[0,\tau)$ in G. The local absolute continuity of $x(\cdot)$ means that it is absolutely continuous on any finite interval $[0,T] \subset [0,\tau)$. A pair $(x(\cdot),u(\cdot))$ is *admissible* if the trajectory $x(\cdot)$ is defined on the entire infinite time interval $[0,\infty)$ in G and the function $t \mapsto e^{-\rho t} g(x(t),u(t))$ is locally integrable on $[0,\infty)$.

Note that, although we do not require the admissible control $u(\cdot)$ or the corresponding discounted utility flow $t \mapsto e^{-\rho t} g(x(t),u(t))$ to be bounded (even locally), by definition the integral

$$J_T(x(\cdot),u(\cdot)) := \int_0^T e^{-\rho t} g(x(t),u(t))\,dt$$

is well-defined for an arbitrary admissible pair $(x(\cdot),u(\cdot))$ and any $T > 0$.

Since the value of the functional (2.1) can be infinite, different optimality concepts can be applied to problem (P) (Carlson, Haurie, Leizarowitz [21]). In the models considered in this paper, the utility functional (2.1) is always bounded from above for all admissible pairs. This allows us to use the standard concept of *strong optimality*. In this case, the integral in (2.1) is understood in the improper sense; i.e., for an arbitrary admissible pair,

$$J(x(\cdot), u(\cdot)) = \lim_{T\to\infty} \int_0^T e^{-\rho t} g(x(t), u(t))\, dt$$

provided that the limit exists. An admissible pair $(x_*(\cdot), u_*(\cdot))$ is called *strongly optimal* (or just *optimal*) in problem (P) if the integral in (2.1) converges (to a finite number) and

$$J(x_*(\cdot), u_*(\cdot)) \geq \limsup_{T\to\infty} \int_0^T e^{-\rho t} g(x(t), u(t))\, dt$$

for any other admissible pair $(x(\cdot), u(\cdot))$.

In models of economic growth, the output $Y(t) := F(x(t), u(t))$ is given by a *production function* $F\colon G \times U \to \mathbb{R}^1$, which is a function of values of the production factors $x(t)$ and control $u(t)$ at $t \geq 0$. We value the output, or a part thereof which is consumed, using the simple logarithmic instantaneous utility function:

$$g(x, u) = \ln F(x, u), \qquad x \in G, \quad u \in U. \tag{2.4}$$

The logarithmic utility function is the simplest case of the isoelastic CRRA function with a constant elasticity of the intertemporal substitution, which is a text-book choice in many economic models (see, for example, Acemoglu [1]). In addition to its simplicity, the logarithmic utility function has another virtue in the context of growth models that is not widely appreciated. Maximizing the aggregate discounted logarithmic utility essentially amounts to maximizing the aggregate discounted growth rates. Consequently, when we seek a welfare-maximizing solution to an economic problem, we actually simultaneously assume that the economy maximizes the aggregate discounted growth rates.

To see the connection, write the corresponding objective functional (2.1) as

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t} \ln F(x(t), u(t))\, dt. \tag{2.5}$$

For an admissible pair $(x(\cdot), u(\cdot))$, the output is given by $Y(t) = F(x(t), u(t))$. Suppose that $Y(\cdot)$ is locally absolutely continuous on $[0, \infty)$, the objective functional (2.1) with instantaneous utility (2.4) converges to a finite number and $\lim_{T\to\infty} e^{-\rho T} \ln Y(T) = 0$. The aggregate discounted value of the growth rate of $Y(\cdot)$ equals

$$\widetilde{J}(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t} \frac{\dot{Y}(t)}{Y(t)}\, dt. \tag{2.6}$$

Integrating the right-hand side of (2.6) by parts yields

$$\widetilde{J}(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t} \frac{d(\ln Y(t))}{dt}\, dt = -\ln Y(0) + \rho \int_0^\infty e^{-\rho t} \ln Y(t)\, dt.$$

Maximizing the goal functional (2.6) is thus equivalent to maximizing the functional (2.5) with the logarithmic instantaneous utility (2.4).

Notice that, in the case of logarithmic instantaneous utility function (2.4), the production function $t \mapsto F(x(t), u(t))$ must be strictly positive. If its values are sufficiently close to 0 for some $t \geq 0$, then the absolute values of the corresponding utility flow $t \mapsto e^{\rho t} \ln F(x(t), u(t))$ can be arbitrarily large.

2.2 The maximum principle and transversality conditions

Define the Hamilton–Pontryagin function $\mathcal{H}\colon [0, \infty) \times G \times U \times \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$ for problem (P) in the usual way:

$$\mathcal{H}(t, x, u, \psi^0, \psi) = \langle \psi, f(x, u) \rangle + \psi^0 e^{-\rho t} g(x, u),$$
$$t \in [0, \infty), \quad x \in G, \quad u \in \mathbb{R}^m, \quad \psi \in \mathbb{R}^n, \quad \psi^0 \in \mathbb{R}^1.$$

In the normal case, when $\psi^0 = 1$, we write $\mathcal{H}(t, x, u, \psi)$ instead of $\mathcal{H}(t, x, u, 1, \psi)$.

The general maximum principle for infinite-horizon optimal control problems states (Theorem 4.2 by Halkin [28]) that if $(x_*(\cdot), u_*(\cdot))$ is a strongly optimal admissible pair in (P), then there exists a nonvanishing pair of adjoint variables $(\psi^0, \psi(\cdot))$, with $\psi^0 \geq 0$ and locally absolutely continuous $\psi(\cdot)\colon [0, \infty) \to \mathbb{R}^n$, that satisfy the core conditions of the maximum principle; i.e., $\psi(\cdot)$ is a solution to the adjoint system

$$\dot{\psi}(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi^0, \psi(t)) \tag{2.7}$$

and the maximum condition holds:

$$\mathcal{H}(t, x_*(t), u_*(t), \psi^0, \psi(t)) \stackrel{\text{a.e.}}{=} \sup_{u \in U} \mathcal{H}(t, x_*(t), u, \psi^0, \psi(t)). \tag{2.8}$$

The general version is valid under rather weak regularity assumptions on the optimal admissible pair $(x_*(\cdot), u_*(\cdot))$ (Aseev and Veliov [13]), but it may not hold in the normal form ($\psi^0 = 0$) and the standard transversality conditions at infinity of the form

$$\lim_{t \to \infty} \psi(t) = 0 \tag{2.9}$$

or

$$\lim_{t \to \infty} \langle x_*(t), \psi(t) \rangle = 0 \tag{2.10}$$

may fail.

Numerous examples demonstrate possible inconsistencies between the transversality conditions (2.9), (2.10) and the core conditions (2.7), (2.8) of the general maximum principle (see Halkin [28] and Section 6 in Aseev and Kryazhimskiy [10]). In this sense the general maximum principle for infinite-horizon optimal control problems is incomplete. In practice, this usually leads to overly wide sets of extremals (trajectories that are suspected to be optimal) distinguished by the core conditions of the maximum principle.

The last decade saw considerable progress towards the development of variants of the maximum principle that involve complementary conditions on the adjoint variable. In the case of autonomous problems with discounting, it was proved that, for a sufficiently high discount rate, the maximum principle holds in the normal form with an adjoint variable specified explicitly by a formula similar to the classical Cauchy formula for the solutions of systems of linear differential equations (Aseev and Kryazhimskiy [9, 10]). This result was extended to several classes of infinite-horizon problems under different growth and regularity assumptions in Aseev, Besov and Kryazhimskiy [8], Aseev and Veliov [11–13] and Besov [20]. The main point is that under certain assumptions the adjoint variable specified by the Cauchy-type formula

$$\psi(t) = Z_*(t) \int_t^{\infty} e^{-\rho t} [Z_*(s)]^{-1} g_x(x_*(s), u_*(s))\, ds, \qquad t \geq 0, \tag{2.11}$$

satisfies the core conditions of the normal-form maximum principle, i.e., conditions (2.7) and (2.8) with $\psi^0 = 1$. Here $(x_*(\cdot), u_*(\cdot))$ is a reference optimal admissible pair in problem (P) and $Z_*(\cdot)$ is the normalized (at $t = 0$) fundamental matrix solution of the linearized system

$$\dot{z}(t) = -\big[f_x(x_*(t), u_*(t))\big]^* z(t). \tag{2.12}$$

The columns of the matrix function $Z_*(\cdot)$ are linearly independent solutions to system (2.12), with $Z_*(0)$ being the identity matrix. The assumed growth conditions ensure that the matrix function $Z_*(\cdot)$ is defined on the whole infinite interval $[0, \infty)$, and that the improper integral on the right-hand side of (2.11) converges absolutely. The validity of the adjoint system (2.7) for function $\psi(\cdot)$ specified by formula (2.11) can be verified directly by differentiation.

The advantage of a maximum principle with an adjoint variable given by (2.11) is that it requires rather weak regularity and growth assumptions, while formula (2.11) is more informative than the standard transversality conditions at infinity and still has a clear economic interpretation (Aseev [5, 6]). It has been shown that, in some problems, relation (2.11) may entail the transversality conditions (2.9) and (2.10) or even stronger pointwise estimates for $\psi(\cdot)$ (see Section 12 in Aseev and Kryazhimskiy [10] as well as Sections 4, 5 in Aseev and Veliov [11]). In other problems, it can be used as an alternative to the standard transversality conditions (see the discussion of Halkin's example in Aseev and Veliov [11]).

The following example illustrates possible inconsistencies between the standard transversality conditions at infinity and the core conditions of the maximum principle.

Example 1 (inconsistency)**.** Consider the following optimal control problem (Q_1):

$$J(S(\cdot), v(\cdot)) = \int_0^{\infty} e^{-\rho t} \ln\big[v(t)S(t)\big]\, dt \to \max,$$

$$\dot{S}(t) = -v(t)S(t), \qquad S(0) = S_0 > 0,$$

$$v(t) > 0.$$

The class of admissible controls for this problem consists of all positive locally integrable functions $v\colon [0, \infty) \to \mathbb{R}^1$. In Section 5, we show directly that the unique optimal solution to this problem is the well-known Hotelling depletion rule $v_*(t) \equiv \rho$ with the corresponding trajectory $S_*(t) = S_0 e^{-\rho t}$, $t \geq 0$.

Let us introduce a new state variable $\widetilde{S}(t) = S(t) + S_1$, $t \geq 0$, where $S_1 > 0$ is a constant. In terms of $\widetilde{S}(\cdot)$, problem (Q_1) is formulated as the following problem $(\widetilde{Q}_1)$:

$$\widetilde{J}(\widetilde{S}(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} \ln\big[v(t)(\widetilde{S}(t) - S_1)\big]\, dt \to \max,$$

$$\dot{\widetilde{S}}(t) = -v(t)(\widetilde{S}(t) - S_1), \qquad \widetilde{S}(0) = S_0 + S_1 > 0,$$

$$v(t) > 0.$$

Since the two problems are equivalent, $v_*(t) \equiv \rho$ must also be a unique optimal admissible control in $(\widetilde{Q}_1)$, with the corresponding trajectory $\widetilde{S}_*(t) = S_*(t) + S_1 = S_0 e^{-\rho t} + S_1$, $t \geq 0$.

We are now able to prove the existence of a unique (up to a positive factor) pair of adjoint variables $\psi^0 = 1$ and $\psi(t) \equiv 1/(\rho S_0)$ associated with the optimal pair $(\widetilde{S}_*(\cdot), v_*(\cdot))$ in $(\widetilde{Q}_1)$ by the core conditions (2.7) and (2.8) of the maximum principle (Theorem 4.2 in Halkin [28]). Then we will see that the transversality conditions (2.9) and (2.10) fail.

For the optimal process $(\widetilde{S}_*(\cdot), v_*(\cdot))$, the adjoint system (2.7) reads

$$\dot{\psi}(t) = -\mathcal{H}_{\widetilde{S}}(t, \widetilde{S}_*(t), v_*(t), \psi^0, \psi(t)) = \rho\psi(t) - \frac{\psi^0}{S_0}.$$

This implies

$$\psi(t) = e^{\rho t}\left(\psi(0) - \frac{\psi^0}{\rho S_0}\right) + \frac{\psi^0}{\rho S_0}, \qquad t \geq 0,$$

where $\psi(0) \in \mathbb{R}^1$ is an initial state of the adjoint variable $\psi(\cdot)$.

According to the maximum condition (2.8), for a.e. $t \in [0, \infty)$ we have

$$-\rho\psi(t) S_0 e^{-\rho t} + \psi^0 e^{-\rho t} \ln \rho = \sup_{v>0}\big(- v\psi(t) S_0 e^{-\rho t} + \psi^0 e^{-\rho t} \ln v\big), \tag{2.13}$$

which implies that $\psi^0 > 0$. (Indeed, if $\psi^0 = 0$, the supremum on the right-hand side of (2.13) would be taken of a linear function of v, in which case the supremum could not be attained in the intermediate point $v = \rho$.)

Given $\psi^0 > 0$, we can scale ψ^0 and $\psi(\cdot)$ by a positive factor to get $\psi^0 = 1$. Then,

$$\psi(t) = e^{\rho t}\left(\psi(0) - \frac{1}{\rho S_0}\right) + \frac{1}{\rho S_0}, \qquad t \geq 0.$$

Substituting this expression for the adjoint variable in the maximum condition (2.13) and setting $\psi^0 = 1$, we see that a unique adjoint variable satisfying all conditions of the maximum principle in the normal form (Theorem 4.2 in Halkin [28]) is given by

$$\psi(t) \equiv \frac{1}{\rho S_0} > 0, \qquad t \geq 0.$$

Thus, $\psi^0 = 1$ and $\psi(\cdot)$, $\psi(t) \equiv 1/(\rho S_0)$, $t \geq 0$, is a unique (up to a positive factor) pair of adjoint variables associated with the optimal pair $(\widetilde{S}_*(\cdot), v_*(\cdot))$ in problem $(\widetilde{Q}_1)$ by the core conditions of the maximum principle. Since $\widetilde{S}_*(t) = S_0 e^{-\rho t} + S_1$, $t \geq 0$, both transversality conditions (2.9) and (2.10) fail. However, the Cauchy-type formula (2.11) for the adjoint variable is still valid (see Example 3 in Aseev [6]).

2.3 Existence of a solution in models with resource constraints

The issue of the existence of solutions to optimal control problems has always been critical and theoretically subtle. In the words of Paul Romer [44], "without such a theorem there is no guarantee that there exists a path which satisfies the necessary conditions or that one of the paths satisfying the necessary conditions will be a solution." It is well known that boundedness of the functions $f(x(\cdot), u(\cdot))$ and $g(x(\cdot), u(\cdot))$ for all admissible pairs $(x(\cdot), u(\cdot))$, as well as closedness of the set U, is essential for standard existence results (Balder [15] and Cesari [23]). The problem is that many economic growth models that involve an exhaustible resource do not ensure boundedness of controls. This makes the direct application of the maximum principle to such problems questionable.

The typical control system describing the process of depletion of an exhaustible resource has the form

$$\dot{S}(t) = -R(t), \qquad S(0) = S_0, \tag{2.14}$$

where $S(t)$ is the stock of an exhaustible resource and $R(t)$ is the rate of its depletion at time $t \geq 0$. The initial value of the resource supply is $S_0 > 0$. The rate $R(\cdot)$ is usually treated as a control. Natural assumptions would be the integrability of the function $R(\cdot)$ on $[0, \infty)$ and the following pointwise and integral constraints:

$$R(t) \geq 0 \tag{2.15}$$

and

$$\int_0^\infty R(t)\, dt \leq S_0. \tag{2.16}$$

In this situation the right-hand side of (2.14) can be unbounded.

The integral constraint (2.16) takes the model beyond the framework of problem (P), leaving open the question of which maximum principle can be applied. Although the integral constraint can be replaced by the state constraint $S(t) \geq 0$, the introduction of state constraints calls for substantially more complicated maximum principles that involve general entities such as functions of bounded variation (Hartl, Sethi and Vickson [30]).

In certain situations, we can remove the integral constraint (2.16) by introducing a new control $v(\cdot)$:

$$v(t) = \frac{R(t)}{S(t)}, \qquad t \geq 0. \tag{2.17}$$

For example, this can be done in the presence of a strict pointwise control constraint

$$R(t) > 0, \qquad t \geq 0. \tag{2.18}$$

Indeed, in this case it follows from (2.14) and (2.16) that $S(t) > 0$ for all $t \geq 0$, so the function $v(\cdot)$ is well defined. Moreover, since $S(\cdot)$ is an absolutely continuous solution to the initial value problem (2.14) corresponding to an admissible control $R(\cdot)$ on $[0, \infty)$, the function $v(\cdot)$ defined by (2.17) is a control and $S(\cdot)$ is an absolutely continuous solution to the initial value problem

$$\dot{S}(t) = -v(t)S(t), \qquad S(0) = S_0. \tag{2.19}$$

System (2.14) with constraints (2.16), (2.18) is thus completely equivalent to system (2.19) subject to the only pointwise constraint

$$v(t) > 0, \qquad t \geq 0. \tag{2.20}$$

However, even in this case the admissible controls $v(\cdot)$ defined by (2.17) are unbounded.

One can also consider system (2.19) with the nonstrict constraint

$$v(t) \geq 0, \qquad t \geq 0, \tag{2.21}$$

instead of (2.20) (its trajectories correspond to trajectories of system (2.14)–(2.16), but not all trajectories of the latter have their counterparts in system (2.19), (2.21)).

Example 2 (nonexistence)**.** Consider the following optimal control problem (Q_2):

$$J(S(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t} v(t) S(t)\, dt \to \max,$$
$$\dot{S}(t) = -v(t)S(t), \qquad S(0) = S_0,$$
$$v(t) \geq 0.$$

The class of admissible controls for the above problem comprises all nonnegative locally integrable functions $v\colon [0,\infty) \to \mathbb{R}^1$. The boundedness of the functional $J(S(\cdot), v(\cdot))$ on the set of all admissible pairs $(S(\cdot), v(\cdot))$ allows us to invoke strong optimality. Since the aim is to optimally deplete a finite initial stock $S_0 > 0$, we essentially have an infinite-horizon cake-eating problem with a linear instantaneous utility function of consumption. Intuition suggests that costless extraction coupled with a positive discount rate must lead to an instantaneous depletion.

To see why there is no optimal control in the class of locally integrable functions, note that if $v(\cdot)$ is an admissible control positive on a set of positive measure and $S(\cdot)$ is the corresponding trajectory, then

$$\int_0^\infty e^{-\rho t} v(t) S(t)\, dt < \int_0^\infty v(t) S(t)\, dt \leq S_0. \tag{2.22}$$

If $v(t) \overset{\text{a.e.}}{=} 0$, then we also have $\int_0^\infty e^{-\rho t} v(t) S(t)\, dt = 0 < S_0$. On the other hand, we can construct a maximizing sequence of admissible controls $\{v_k(\cdot)\}$, namely, $v_k(t) \equiv k$, $k = 1, 2, \dots$, with

$$\int_0^\infty e^{-\rho t} v_k(t) S_k(t)\, dt = \int_0^\infty k S_0 e^{-(\rho+k)t}\, dt = \frac{k S_0}{\rho + k} \to S_0 \qquad \text{as} \quad k \to \infty.$$

The supremum of $J(S(\cdot), v(\cdot))$ over all admissible pairs equals S_0, a value that cannot be attained in view of (2.22). Thus, an optimal admissible control in the class of integrable functions does not exist.

Despite the nonexistance of a solution, a formal application of the maximum principle (Halkin [28, Theorem 4.2]) to this problem yields a unique admissible control $v_*(t) \overset{\text{a.e.}}{=} 0$, which satisfies the core conditions (2.7) and (2.8) with the adjoint variables $\psi^0 = 1$ and $\psi(t) \equiv 1$, $t \geq 0$.

Indeed, assume that there is an optimal admissible pair $(S_*(\cdot), v_*(\cdot))$ in (Q_2), and let $(\psi^0, \psi(\cdot))$ be the corresponding pair of adjoint variables. Then

$$\mathcal{H}(t, S, v, \psi^0, \psi) = \big(\psi^0 e^{-\rho t} - \psi\big) vS, \qquad t \geq 0,\ v \geq 0,\ S > 0.$$

In this case, the adjoint system (2.7) and the maximum condition (2.8) read

$$\dot\psi(t) = -\big(\psi^0 e^{-\rho t} - \psi(t)\big) v_*(t),$$
$$\big(\psi^0 e^{-\rho t} - \psi(t)\big) v_*(t) \overset{\text{a.e.}}{=} \sup_{v \geq 0} \big\{ \big(\psi^0 e^{-\rho t} - \psi(t)\big) v \big\}.$$

The supremum of a linear function can only be finite if it is zero. Thus, the right-hand side of the adjoint system vanishes for a.e. $t \geq 0$. Hence, $\psi(t) \equiv \psi(0)$, $t \geq 0$. Then the adjoint system takes the form $0 \overset{\text{a.e.}}{=} -(\psi^0 e^{-\rho t} - \psi(0)) v_*(t)$, which yields $v_*(t) = 0$ for a.e. $t \geq 0$ (since either $\psi_0 \neq 0$ or $\psi(0) \neq 0$).

Thus, the formal application of the general version of the maximum principle to problem (Q_2) yields a unique admissible control $v_*(t) \overset{\text{a.e.}}{=} 0$, $t \geq 0$, which obviously satisfies the adjoint system and the maximum condition with the adjoint variables $\psi^0 = 1$ and $\psi(t) \equiv 1$ (for example), $t \geq 0$. However, there is no optimal admissible control in (Q_2).

3 The DHSS model

The basic DHSS model studied in this paper can be presented in terms of output produced using two types of capital: produced capital and the exhaustible resource. At every instant $t \in [0, \infty)$, the economy produces output $Y(t) > 0$ described by a Cobb–Douglas function:

$$Y(t) = K(t)^{\varkappa} R(t)^{\gamma} \qquad \text{where} \quad 0 \leq \varkappa \leq 1, \quad 0 < \gamma \leq 1. \tag{3.1}$$

Here, $K(t) > 0$ is the stock of produced capital and $R(t) > 0$ is the speed of resource extraction from the current resource stock $S(t)$ at instant $t \geq 0$. The output can be either consumed, generating utility, or invested in increasing the stock of produced capital, which deteriorates with time. If a part $u(t) \in [0, 1)$ of the output $Y(t)$ is invested, the amount of produced capital available at time t varies according to the rule

$$\dot K(t) = u(t) K(t)^{\varkappa} R(t)^{\gamma} - \delta K(t) \qquad \text{where} \quad \delta \geq 0. \tag{3.2}$$

This makes the produced capital renewable at the expense of consumption. The stock of produced capital depreciates at a constant rate δ. The initial stock of produced capital is $K(0) = K_0 > 0$. The remaining output, $(1 - u(t))Y(t)$, $t \geq 0$, is consumed.

The resource, on the other hand, is nonrenewable. The finite resource stock imposes the following integral constraint on the depletion speed $R(\cdot)$:

$$\int_0^\infty R(t)\, dt \leq S_0, \tag{3.3}$$

where $S_0 > 0$ is the initial resource stock.

The problem facing the economy is how to optimally deplete the resource and invest in produced capital. Welfare is measured by a discounted logarithmic utility function of consumption. As we have argued at the end of Section 2.1, maximizing the aggregate discounted logarithmic utility essentially amounts to maximizing the aggregate discounted growth rates. This leads to the following objective functional for the economy (see (3.1)):

$$\begin{aligned} J_1(K(\cdot), u(\cdot), R(\cdot)) &= \int_0^\infty e^{-\rho t} \ln[(1-u(t))Y(t)]\, dt \\ &= \int_0^\infty e^{-\rho t} \big[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\big]\, dt, \end{aligned}$$

where $\rho > 0$ is a discount rate.

Setting $\varkappa = 0$ leads to a *pure depletion* economy. In a pure depletion economy the only question is how to optimally deplete the resource, and we briefly discuss this special case before turning to the full model. We ignore the case $\gamma = 0$ as it implies the irrelevance of the resource. Benchekroun and Withagen [19] provide an explicit solution to the model for $\varkappa + \gamma = 1$ and $\delta = 0$. This parametrization implies constant returns to scale in the production of the final consumption good. In this paper we focus on the general cases of any returns to scale.

Given parameters $0 \le \varkappa \le 1$, $0 < \gamma \le 1$, $\delta \ge 0$, $\rho > 0$, $K_0 > 0$ and $S_0 > 0$, the optimization problem $J(K(\cdot), u(\cdot), R(\cdot)) \to \max$, subject to equation (3.2) and constraint (3.3), can be formulated as an infinite-horizon optimal control problem (P_1):

$$J_1(K(\cdot), u(\cdot), R(\cdot)) = \int_0^\infty e^{-\rho t} \big[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\big]\, dt \to \max, \tag{3.4}$$

$$\dot{K}(t) = u(t) K(t)^{\varkappa} R(t)^{\gamma} - \delta K(t), \qquad K(0) = K_0 > 0, \tag{3.5}$$

$$u(t) \in [0,1), \qquad R(t) > 0, \qquad \int_0^\infty R(t)\, dt \le S_0. \tag{3.6}$$

By an admissible control $w(\cdot)\colon [0,\infty) \to \mathbb{R}^2$ in problem (P_1) we mean a pair $w(\cdot) = (u(\cdot), R(\cdot))$ comprising a measurable function $u(\cdot)$ and an integrable function $R(\cdot)$, each of which is defined on the infinite time interval $[0,\infty)$ and satisfies the constraints in (3.6). In view of (3.5) and (3.6), for any admissible control $w(\cdot) = (u(\cdot), R(\cdot))$ the corresponding admissible trajectory $K(\cdot)$ always exists on the whole infinite interval $[0,\infty)$.

It can be shown that for any admissible pair $(K(\cdot), w(\cdot))$ the improper integral in (3.4) either converges to a finite real or diverges to $-\infty$. Moreover, it is uniformly bounded from above. In other words, there is a number $M \ge 0$ such that

$$\sup_{(K(\cdot), w(\cdot))} \int_0^\infty e^{-\rho t} \big[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\big]\, dt \le M, \tag{3.7}$$

where the supremum is taken over all admissible pairs $(K(\cdot), w(\cdot))$.

Indeed, using the trivial inequalities $\ln R(t) < R(t)$, $R(t)^\gamma < R(t) + 1$ and $\dot{K}(t) < (K(t)+1)(R(t)+1)$, we see that

$$\ln(K(t)+1) < \ln(K_0+1) + S_0 + t, \qquad t \ge 0, \tag{3.8}$$

and

$$\int_T^{T'} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt$$
$$< \omega(T) := \left[\frac{\ln(K_0+1)+S_0+T}{\rho} + \frac{1}{\rho^2} + S_0\right] e^{-\rho T} \quad (3.9)$$

for any $0 \le T < T'$. Further, we have

$$\int_0^{T'} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt$$
$$= \int_0^{T} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt$$
$$+ \int_T^{T'} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt.$$

Due to (3.9), for any $T \ge 0$ this implies

$$\limsup_{T'\to\infty} \int_0^{T'} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt$$
$$\le \int_0^{T} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt + \omega(T).$$

Hence,

$$\limsup_{T\to\infty} \int_0^{T} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt$$
$$\le \liminf_{T\to\infty} \int_0^{T} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt.$$

Since the reverse inequality is always true, the limit

$$J_1(K((\cdot)), u((\cdot)), R((\cdot))) = \lim_{T\to\infty} \int_0^{T} e^{-\rho t}\bigl[\ln(1-u(t)) + \varkappa \ln K(t) + \gamma \ln R(t)\bigr]\,dt$$

exists and equals either a finite number or $-\infty$ due to (3.9).

Thus, for any admissible pair $(K(\cdot), w(\cdot))$ the improper integral in (3.4) either converges to a finite real or diverges to $-\infty$, while inequality (3.9) implies (3.7). This fact allows us to understand the optimality of an admissible pair $(K_*(\cdot), w_*(\cdot))$ in problem (P_1) in the strong sense.

4 Reduction to a one-dimensional problem without integral constraints

Let us introduce a new state variable $x(\cdot)\colon [0,\infty) \to \mathbb{R}^1$ and a new control variable $v(\cdot)\colon [0,\infty) \to [0,\infty)$ as follows:

$$x(t) = \frac{K(t)^{1-\varkappa}}{S(t)^{\gamma}}, \qquad v(t) = \frac{R(t)}{S(t)}, \qquad t \ge 0. \quad (4.1)$$

Here, as in Section 2, the state variable $S(\cdot)$ represents the current supply of the exhaustible resource. This variable is a (Carathéodory) solution to the initial value problem

$$\dot{S}(t) = -R(t), \qquad S(0) = S_0, \tag{4.2}$$

for a given admissible control $R(\cdot)$ on $[0, \infty)$ (see (2.14)). So the control variable $v(\cdot)$ represents the extraction rate of the resource. Due to the pointwise and integral constraints on $R(\cdot)$ in (3.6), we have $S(t) > 0$ for all $t > 0$. Thus, the quantities $x(t)$ and $v(t)$ are well defined for all $t > 0$. Moreover, the function $v(\cdot)$ is locally integrable since $R(\cdot)$ is integrable and $S(\cdot)$ is positive and continuous.

According to the production function (3.1), the quantity $Y(K(t), S(t))$ is the instantaneous output that can be attained by the usage of current stocks $K(t)$ and $S(t)$ of the capital and the resource at time $t \geq 0$. The state variable $x(\cdot)$ thus represents a hypothetical current capital coefficient:

$$x(t) = \frac{K(t)}{Y(K(t), S(t))} = \frac{K(t)^{1-\varkappa}}{S(t)^{\gamma}}, \qquad t \geq 0.$$

Since $x(\cdot)$ is a (locally) absolutely continuous function, we can calculate its derivative a.e. on $[0, \infty)$:

$$\begin{aligned}
\dot{x}(t) &= (1-\varkappa)\frac{K(t)^{-\varkappa}\dot{K}(t)}{S(t)^{\gamma}} - \gamma\frac{K(t)^{1-\varkappa}\dot{S}(t)}{S(t)^{1+\gamma}} \\
&= (1-\varkappa)\frac{K(t)^{-\varkappa}\left(u(t)K(t)^{\varkappa}R(t)^{\gamma} - \delta K(t)\right)}{S(t)^{\gamma}} + \gamma\frac{K(t)^{1-\varkappa}v(t)S(t)}{S(t)^{1+\gamma}} \\
&= (1-\varkappa)u(t)v(t)^{\gamma} + \left(\gamma v(t) - (1-\varkappa)\delta\right)x(t).
\end{aligned}$$

Thus, $x(\cdot)$ is a Carathéodory solution to the differential equation

$$\dot{x}(t) = (1-\varkappa)u(t)v(t)^{\gamma} + \big(\gamma v(t) - (1-\varkappa)\delta\big)x(t), \qquad t > 0, \tag{4.3}$$

satisfying the initial condition

$$x(0) = x_0 = \frac{K_0^{1-\varkappa}}{S_0^{\gamma}}. \tag{4.4}$$

Now we express the functional $J_1(K(\cdot), u(\cdot), R(\cdot))$ (see (3.4)) in terms of the variables $x(\cdot)$, $u(\cdot)$ and $v(\cdot)$. Consider the second term in the integrand in (3.4):

$$\int_0^{\infty} e^{-\rho t}\ln K(t)\,dt = \frac{\ln K_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}\frac{\dot{K}(t)}{K(t)}\,dt \tag{4.5}$$

(it is not difficult to check that $e^{-\rho T}\ln K(T) \to 0$ as $T \to +\infty$ by virtue of (3.5) and (3.8)).

Substituting $\dot{K}(t)$ from (3.5) into (4.5), we obtain

$$\begin{aligned}
\int_0^{\infty} e^{-\rho t}\ln K(t)\,dt &= \frac{\ln K_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}\frac{u(t)K(t)^{\varkappa}R(t)^{\gamma} - \delta K(t)}{K(t)}\,dt \\
&= \frac{\ln K_0}{\rho} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}\left[u(t)v(t)^{\gamma}\frac{S(t)^{\gamma}}{K(t)^{1-\varkappa}} - \delta\right]dt \\
&= \frac{\ln K_0}{\rho} - \frac{\delta}{\rho^2} + \frac{1}{\rho}\int_0^{\infty} e^{-\rho t}\frac{u(t)v(t)^{\gamma}}{x(t)}\,dt.
\end{aligned} \tag{4.6}$$

Similarly,

$$\begin{aligned}\int_0^T e^{-\rho t}\ln R(t)\,dt &= \int_0^T e^{-\rho t}\big[\ln v(t)+\ln S(t)\big]\,dt\\ &= \int_0^T e^{-\rho t}\ln v(t)\,dt + \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \frac{1}{\rho}\int_0^T e^{-\rho t}\frac{\dot S(t)}{S(t)}\,dt\\ &= \frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \int_0^T e^{-\rho t}\left[\ln v(t) - \frac{v(t)}{\rho}\right]dt. \qquad (4.7)\end{aligned}$$

To get rid of the term $e^{-\rho T}\ln S(T)$ in (4.7), let us first introduce for arbitrary $S_0 > 0$ the quantity

$$M(S_0) = \sup_{R(\cdot)} \int_0^\infty e^{-\rho t}\ln R(t)\,dt,$$

where the supremum is taken over all admissible controls $R(\cdot)$ in problem (P_1).

Clearly, the trivial estimate $\ln R(t) < R(t)$ and the integral constraint in (3.6) yield the bound $M(S_0) \le S_0$, and so for any $S_0 > 0$ the number $M(S_0)$ is well defined and finite. Moreover, representing $R(t)$ as $S_0\widetilde{R}(t)$ for $t \ge 0$ (with $\int_0^\infty \widetilde{R}(t)\,dt \le 1$), we see that

$$M(S_0) = \int_0^\infty e^{-\rho t}\ln S_0\,dt + M(1) = \frac{\ln S_0}{\rho} + M(1).$$

Therefore, for any admissible control $R(\cdot)$ and any $T \ge 0$,

$$\int_T^\infty e^{-\rho t}\ln R(t)\,dt \le e^{-\rho T}M(S(T)) = e^{-\rho T}\left(\frac{\ln S(T)}{\rho} + M(1)\right). \qquad (4.8)$$

If $e^{-\rho T}\ln S(T) \to 0$ as $T \to +\infty$, then, passing to the limit as $T \to +\infty$ in (4.7), we obtain

$$\int_0^\infty e^{-\rho t}\ln R(t)\,dt = \frac{\ln S_0}{\rho} + \int_0^\infty e^{-\rho t}\left[\ln v(t) - \frac{v(t)}{\rho}\right]dt, \qquad (4.9)$$

where both sides may be $-\infty$.

If $e^{-\rho T}\ln S(T)$ does not tend to zero as $T \to +\infty$, i.e.,

$$\liminf_{T\to+\infty} e^{-\rho T}\ln S(T) < 0,$$

then from (4.8) we have

$$\liminf_{T\to+\infty} \int_T^\infty e^{-\rho t}\ln R(t)\,dt < 0.$$

This is only possible when

$$\int_0^\infty e^{-\rho t}\ln R(t)\,dt = \lim_{T\to\infty}\int_0^T e^{-\rho t}\ln R(t)\,dt = -\infty$$

(for the same reasons as above, the limit exits and is either a finite real or $-\infty$). Since

$$\limsup_{T\to+\infty} e^{-\rho T}\ln S(T) \le 0,$$

the last integral in (4.7) also tends to $-\infty$ in this case. Hence, (4.9) holds in this case as well.

Thus, neglecting constant terms, we arrive at the functional

$$\widetilde{J}_1(x(\cdot),u(\cdot),v(\cdot)) = \int_0^\infty e^{-\rho t}\left\{\ln(1-u(t)) + \gamma\left[\ln v(t) - \frac{v(t)}{\rho}\right] + \frac{\varkappa}{\rho}\frac{u(t)v(t)^\gamma}{x(t)}\right\}dt. \quad (4.10)$$

Now consider the following optimal control problem (see (4.3), (4.4) and (4.10)) $(\widetilde{P}_1)$:

$$\widetilde{J}_1(x(\cdot),u(\cdot),v(\cdot)) = \int_0^\infty e^{-\rho t}\left\{\ln(1-u(t)) + \gamma\left[\ln v(t) - \frac{v(t)}{\rho}\right] + \frac{\varkappa}{\rho}\frac{u(t)v(t)^\gamma}{x(t)}\right\}dt \to \max, \quad (4.11)$$

$$\dot{x}(t) = (1-\varkappa)u(t)v(t)^\gamma + \big(\gamma v(t) - (1-\varkappa)\delta\big)x(t), \qquad x(0) = x_0, \quad (4.12)$$

$$v(t) \in (0,\infty), \qquad u(t) \in [0,1). \quad (4.13)$$

We say that a control $\tilde{w}(\cdot) = (u(\cdot),v(\cdot))\colon [0,\infty) \to [0,1)\times(0,\infty)$ (which is a pair of measurable functions) is admissible in problem $(\widetilde{P}_1)$ if the functions

$$t \mapsto v(t) \qquad \text{and} \qquad t \mapsto e^{-\rho t}\left\{\ln(1-u(t)) + \gamma\left[\ln v(t) - \frac{v(t)}{\rho}\right] + \frac{\varkappa}{\rho}\frac{u(t)v(t)^\gamma}{x(t)}\right\}$$

are locally integrable on $[0,\infty)$. The corresponding trajectory $x(\cdot)$ is obviously defined on the whole infinite time interval $[0,\infty)$. A pair $(x(\cdot),\tilde{w}(\cdot))$, where $\tilde{w}(\cdot)$ is an admissible control and $x(\cdot)$ is the corresponding trajectory, is called an admissible pair or a process in problem $(\widetilde{P}_1)$.

As shown above, problem $(\widetilde{P}_1)$ is equivalent to problem (P_1) in the following sense:

Lemma 1. *For fixed K_0 and S_0 there is a one-to-one correspondence between processes $(K(\cdot),w(\cdot))$ in problem (P_1) and $(x(\cdot),\tilde{w}(\cdot))$ in problem $(\widetilde{P}_1)$. Moreover, the corresponding values of the objective functionals $J_1(K(\cdot),u(\cdot),R(\cdot))$ and $\widetilde{J}_1(x(\cdot),u(\cdot),v(\cdot))$ are related by a linear transformation of the form*

$$\widetilde{J}_1(x(\cdot),u(\cdot),v(\cdot)) = J_1(K(\cdot),u(\cdot),R(\cdot)) + C_0, \quad (4.14)$$

where C_0 depends only on δ, ρ, $\varkappa$, γ, K_0 and S_0.

We will also need another equivalent form of the functional in problem $(\widetilde{P}_1)$ for $\varkappa < 1$. Namely, note that

$$\ln K(t) = \frac{1}{1-\varkappa}\ln x(t) + \frac{\gamma}{1-\varkappa}\ln S(t), \qquad \ln R(t) = \ln v(t) + \ln S(t), \qquad t \geq 0,$$

for $\varkappa < 1$ (see (4.1)), and so

$$\int_0^T e^{-\rho t}\big[\varkappa \ln K(t) + \gamma \ln R(t)\big]\,dt = \int_0^T e^{-\rho t}\left[\frac{\varkappa}{1-\varkappa}\ln x(t) + \frac{\gamma}{1-\varkappa}\ln S(t) + \gamma\ln v(t)\right]dt$$

$$= \frac{\gamma}{1-\varkappa}\frac{\ln S_0 - e^{-\rho T}\ln S(T)}{\rho} + \int_0^T e^{-\rho t}\left[\frac{\varkappa}{1-\varkappa}\ln x(t) + \gamma\ln v(t) - \frac{\gamma v(t)}{(1-\varkappa)\rho}\right]dt. \quad (4.15)$$

For the same reasons as in (4.7), either $e^{-\rho T}\ln S(T) \to 0$ or all integrals in (4.15) tend to $-\infty$ as $T \to \infty$. Therefore,

$$\widetilde{J}_1(x(\cdot),u(\cdot),v(\cdot)) = C_1 + \int_0^\infty e^{-\rho t}\biggl\{\ln(1-u(t)) + \gamma\ln v(t) - \frac{\gamma v(t)}{(1-\varkappa)\rho} + \frac{\varkappa}{1-\varkappa}\ln x(t)\biggr\}dt \tag{4.16}$$

for $\varkappa < 1$, where C_1 depends only on δ, ρ, $\varkappa$, γ, K_0 and S_0.

Note that, structurally, problem $(\widetilde{P}_1)$ is simpler than problem (P_1), because problem $(\widetilde{P}_1)$ does not contain integral constraints on the control variables. Lemma 1 allows us to deal with problem $(\widetilde{P}_1)$ instead of (P_1). However, the analysis of problem $(\widetilde{P}_1)$ is still hindered by the unboundedness of the set of admissible values of the control $v(\cdot)$. To reduce the set of admissible controls among which it makes sense to seek optimal ones, we make an observation that follows from the form (4.10) of the functional.

Namely, in the case of $\gamma = 1$ we define a function $V_1(\cdot)\colon (0,+\infty) \to (0,+\infty]$ as

$$V_1(x) = \begin{cases} +\infty, & 0 < x \le \varkappa, \\ \dfrac{\rho x}{x-\varkappa}, & \varkappa < x. \end{cases}$$

In the case of $\gamma < 1$, let $V_1(\cdot)$ be defined as

$$V_1(x) = \Bigl(\rho + \frac{\varkappa}{x}\Bigr)^{1/(1-\gamma)} + 1, \qquad x > 0.$$

In the latter case, we also introduce a function $V_2(\cdot)\colon [0,+\infty) \to (0,+\infty)$,

$$V_2(t) = \biggl(\Bigl(\rho + \frac{\varkappa}{x_0}\Bigr)^{1/(1-\gamma)} + 1\biggr)e^{(1-\varkappa)\delta t/(1-\gamma)}, \qquad t \ge 0. \tag{4.17}$$

For $\gamma = 1$, the function $V_2(\cdot)$ can be thought of as the identical $+\infty$.

Lemma 2. *Problem $(\widetilde{P}_1)$ is equivalent to each of the problems $(\widetilde{P}_1^i)$, $i = 1,2$, obtained from $(\widetilde{P}_1)$ by imposing the following additional constraint on the control $v(\cdot)$:*

$$v(t) \le V_1(x(t)) \quad (i=1) \qquad \textit{or} \qquad v(t) \le V_2(t) \quad (i=2), \qquad t \ge 0. \tag{4.18}$$

More precisely, for each $i = 1,2$, an admissible process is optimal in problem $(\widetilde{P}_1)$ if and only if it is admissible (i.e., satisfies (4.18)) and optimal in problem $(\widetilde{P}_1^i)$. Moreover, for any admissible process $(x(\cdot),u(\cdot),v(\cdot))$ in problem $(\widetilde{P}_1)$, the process $(\bar{x}(\cdot),u(\cdot),\bar{v}(\cdot))$, where $\bar{v}(t) = \min\{v(t),V_1(x(t))\}$, $t \ge 0$, and $\bar{x}(\cdot)$ is the trajectory corresponding to $(u(\cdot),\bar{v}(\cdot))$, is admissible in problem $(\widetilde{P}_1^i)$, $i = 1,2$, and $\widetilde{J}_1(\bar{x}(\cdot),u(\cdot),\bar{v}(\cdot)) \ge \widetilde{J}_1(x(\cdot),u(\cdot),v(\cdot))$.

Proof. Let us first compare problems $(\widetilde{P}_1)$ and $(\widetilde{P}_1^1)$ (i.e., let $i = 1$). For any admissible process $(x(\cdot),u(\cdot),v(\cdot))$ in problem $(\widetilde{P}_1)$, we have $\bar{v}(t) \le v(t)$, $t \ge 0$, so $\bar{x}(t) \le x(t)$, $t \ge 0$ (see (4.12)), and hence $V_1(\bar{x}(t)) \ge V_1(x(t))$, $t \ge 0$. Therefore, $(\bar{x}(\cdot),u(\cdot),\bar{v}(\cdot))$ is admissible in problem $(\widetilde{P}_1^1)$. Checking directly that the derivative of the integrand in (4.10) with respect to $v(t)$ is negative for $v(t) \ge V_1(x(t))$, we find that

$$\widetilde{J}_1(x(\cdot),u(\cdot),v(\cdot)) \le \widetilde{J}_1(x(\cdot),u(\cdot),\bar{v}(\cdot)) \le \widetilde{J}_1(\bar{x}(\cdot),u(\cdot),\bar{v}(\cdot)) \tag{4.19}$$

(here $(x(\cdot), u(\cdot), \bar{v}(\cdot))$ is not an admissible process, and by $\widetilde{J}_1(x(\cdot), u(\cdot), \bar{v}(\cdot))$ we mean the value calculated by formula (4.10)). Moreover, the first inequality in (4.19) is strict if $\bar{v}(t) \neq v(t)$ on a set of positive measure of values of t. This completes the proof for $i = 1$. To handle the case $i = 2$, note that $x(t) \geq x_0 e^{-(1-\varkappa)\delta t}$, $t \geq 0$ (see (4.12)), and so $V_1(x(t)) < V_2(t)$, $t \geq 0$, for any admissible trajectory $x(\cdot)$. This implies that the set of admissible processes in problem $(\widetilde{P}_1^2)$ is intermediate between those in problems $(\widetilde{P}_1)$ and $(\widetilde{P}_1^1)$. Then the assertion for $(\widetilde{P}_1^2)$ follows from the assertion for $(\widetilde{P}_1^1)$. $\square$

We start our analysis of problem $(\widetilde{P}_1)$ by considering a special case $\varkappa = 0$ of pure resource depletion.

5 Pure resource depletion ($\varkappa = 0$)

In the case $\varkappa = 0$, problem $(\widetilde{P}_1)$ (see (4.11)–(4.13)) takes the following form:

$$\widetilde{J}_1(x(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t}\left\{\ln(1 - u(t)) + \gamma\left[\ln v(t) - \frac{v(t)}{\rho}\right]\right\} dt \to \max,$$

$$\dot{x}(t) = u(t)v(t)^\gamma + (\gamma v(t) - \delta)x(t), \qquad x(0) = x_0,$$

$$v(t) \in (0, \infty), \qquad u(t) \in [0, 1).$$

One can observe that the integrand in the functional above does not depend on the state variable $x(\cdot)$. Therefore, since the controls $u(\cdot)$ and $v(\cdot)$ are independent, the solution of problem $(\widetilde{P}_1)$ can be obtained by directly maximizing the integrand with respect to the variables $u(t)$ and $v(t)$ for every fixed $t \geq 0$:

$$u_*(t) \equiv 0 \qquad \text{and} \qquad v_*(t) \equiv \rho, \qquad t \geq 0,$$

and the corresponding optimal trajectory is

$$x_*(t) = x_0 e^{(\gamma\rho - \delta)t}, \qquad t \geq 0.$$

Thus, the optimal extraction policy $v_*(\cdot)$ follows the Hotelling rule (Hotelling [33]), while the optimal investment policy $u_*(\cdot)$ vanishes.

In terms of the initial state variables $K(\cdot)$ and $S(\cdot)$ (see (3.5) and (4.2)) the corresponding optimal capital stock $K_*(\cdot)$ and optimal resource stock $S_*(\cdot)$ in problem (P_1) are $K_*(t) = K_0 e^{-\delta t}$ and $S_*(t) = S_0 e^{-\rho t}$, $t \geq 0$. Accordingly, $R_*(t) = \rho S_*(t) = \rho S_0 e^{-\rho t}$, $t \geq 0$.

It is easy to see that in the case of $\varkappa = 0$, $\gamma = 1$ and $\delta = 0$, problem (P_1) coincides with problem (Q_1) discussed above in Example 1. Hence, $S_*(t) = S_0 e^{-\rho t}$ and $v_*(t) \equiv \rho$, $t \geq 0$, is a unique optimal admissible pair in problem (Q_1). Accordingly, $\widetilde{S}_*(t) = S_0 e^{-\rho t} + S_1$ and $v_*(t) \equiv \rho$, $t \geq 0$, is a unique optimal admissible pair in $(\widetilde{Q}_1)$.

Our solution of problem (P_1) in the case $\varkappa = 0$ is based on the fact that the integrand in the functional in the reduced problem $(\widetilde{P}_1)$ does not depend on the state variable $x(\cdot)$. This allows one to solve problem $(\widetilde{P}_1)$ by direct maximization of the integrand for each

$t \geq 0$ without appealing to any existence result and necessary conditions for optimality. However, such an approach does not apply if $\varkappa > 0$. In this case, we need a more systematic study based on proving an existence result and then applying an appropriate version of the Pontryagin maximum principle. Thus, in the next section we formulate a general result on the existence of an optimal process in problem (P_1), and then we prove it in the most realistic case $\gamma < 1$ of weak scale effects in the resource use in production. The special knife-edge case $\gamma = 1$ will be studied separately in Section 8.

Everywhere below we assume $\varkappa > 0$.

6 Existence result

In this section, we present a general result on the existence of an optimal process in the series of equivalent problems formulated in Sections 3 and 4 and prove it in the case $0 < \gamma < 1$ (assertion (i)). The proof in the knife-edge case $\gamma = 1$ (assertion (ii)) will be given below in Section 8.

Theorem 1. (i) *In the case of $0 < \gamma < 1$, in problem $(\widetilde{P}_1^2)$ (and hence in problems $(\widetilde{P}_1)$, $(\widetilde{P}_1^1)$ and (P_1) as well) there exists an optimal process for any initial value $x_0 > 0$ (for any $K_0, S_0 > 0$).*

(ii) *In the case of $\gamma = 1$ and $\varkappa + \rho/\delta \geq 1$, an optimal process in problem $(\widetilde{P}_1^2) = (\widetilde{P}_1)$ (and hence in problems $(\widetilde{P}_1^1)$ and (P_1) as well) exists if $x_0 \geq \varkappa$ (i.e., $K_0^{1-\varkappa} \geq \varkappa S_0$) and does not exist if $x_0 < \varkappa$ (i.e., $K_0^{1-\varkappa} < \varkappa S_0$).*

In the case of $\gamma = 1$ and $\varkappa + \rho/\delta < 1$, an optimal process in problem $(\widetilde{P}_1^2) = (\widetilde{P}_1)$ (and hence in problems $(\widetilde{P}_1^1)$ and (P_1) as well) exists if $x_0 \geq \underline{x} := \varkappa(1-\varkappa)\rho/((1-\varkappa)\delta - \varkappa\rho)$ (i.e., $K_0^{1-\varkappa} \geq \underline{x} S_0$) and does not exist if $x_0 < \underline{x}$ (i.e., $K_0^{1-\varkappa} < \underline{x} S_0$).

Proof of assertion (i) *of Theorem* 1. Let $0 < \gamma < 1$. The easiest way to justify the existence is to refer to Theorem 1 in Besov [20][1]. However, in order to apply Theorem 1 in Besov [20], we need to reduce our problem to a form with uniformly bounded controls. To this end, we introduce the additional deterministic state variable $V_2(\cdot)$ (see (4.17)) and replace the control $v(\cdot)$ by $V_2(\cdot)\tilde{v}(\cdot)$ in all relations of problem $(\widetilde{P}_1^2)$. We also denote $u(\cdot)\tilde{v}(\cdot)^\gamma$ by $\tilde{u}(\cdot)$. We then arrive at the following equivalent problem $(\widetilde{P}_2)$:

$$\widetilde{J}_2(x(\cdot), V_2(\cdot), \tilde{u}(\cdot), \tilde{v}(\cdot))$$

$$= \int_0^\infty e^{-\rho t} \left\{ \ln\bigl(\tilde{v}(t)^\gamma - \tilde{u}(t)\bigr) + \gamma \left[\ln V_2(t) - \frac{\tilde{v}(t) V_2(t)}{\rho} \right] + \frac{\varkappa}{\rho} \frac{\tilde{u}(t) V_2(t)^\gamma}{x(t)} \right\} dt \to \max, \quad (6.1)$$

$$\dot{x}(t) = (1-\varkappa)\tilde{u}(t)V_2(t)^\gamma + \bigl(\gamma V_2(t)\tilde{v}(t) - (1-\varkappa)\delta\bigr)x(t), \qquad x(0) = x_0,$$

$$\dot{V}_2(t) = \frac{(1-\varkappa)\delta}{1-\gamma} V_2(t), \qquad V_2(0) = \left(\rho + \frac{\varkappa}{x_0}\right)^{1/(1-\gamma)} + 1,$$

$$(\tilde{u}(t), \tilde{v}(t)) \in U := \{(u, v) \in \mathbb{R}^2 \colon 0 \leq u < v^\gamma \leq 1\}.$$

[1]Alternatively, the standard existence result of Balder [15] can be used, but his assumptions are slightly more cumbersome.

Let us check all hypotheses of Theorem 1 in Besov [20] for problem $(\widetilde{P}_2)$.

Conditions (A_g) and (A1) from Besov [20] obviously hold.

The set U is convex, the dynamics are linear in control, and the only nonlinear term in the instantaneous utility function is $\ln(\tilde{v}^\gamma - \tilde{u})$, which is a concave function on $U \ni (\tilde{u}, \tilde{v})$. Hence, condition (A2) in Besov [20] follows.

Finally, condition (A3) in Besov [20] is a uniform upper bound for finite parts of the tail of the functional (weakening of the strong uniform integrability condition used by Balder [15]). Clearly, when proving such a bound, we can use the form (4.10) of the functional, because it differs from (6.1) by notation only. In (4.10) we have $\ln(1-u(t)) \le 0$, $\ln v(t) - v(t)/\rho \le \ln\rho - 1$, and so we only need to estimate the integral of the function $e^{-\rho t}\varkappa\rho^{-1}u(t)v(t)^\gamma/x(t)$. To this end, we integrate by parts:

$$\int_T^{T'} e^{-\rho t}\frac{\varkappa}{\rho}\frac{u(t)v(t)^\gamma}{x(t)}\,dt = \frac{\varkappa}{\rho}\int_T^{T'} e^{-\rho t}\left[\frac{\dot{K}(t)}{K(t)} + \delta\right]dt$$

$$= \varkappa\int_T^{T'} e^{-\rho t}\left[\ln K(t) + \frac{\delta}{\rho}\right]dt + \frac{\varkappa}{\rho}\left[e^{-\rho T'}\ln K(T') - e^{-\rho T}\ln K(T)\right] \to 0 \quad (6.2)$$

as $T, T' \to \infty$, due to (3.8) and the trivial bound $K(t) \ge K_0 e^{-\delta t}$. This estimate completes the verification of the hypotheses of Theorem 1 in Besov [20], and so assertion (i) is proved. $\square$

Now we are ready to characterize all optimal processes in the case $0 < \gamma < 1$.

7 Weak scale effects in resource use ($0 < \gamma < 1$)

First we consider the most realistic case $0 < \varkappa < 1$, and then the special case $\varkappa = 1$.

7.1 Weak scale effects in both production factors ($0 < \gamma < 1$ and $0 < \varkappa < 1$)

Since $0 < \gamma < 1$, Theorem 1(i) guarantees the existence of an optimal process in problem $(\widetilde{P}_1)$. Further, since $0 < \varkappa < 1$, we can employ the equivalent (up to a constant) form (4.16) of the functional and set $\varkappa' := 1 - \varkappa > 0$ to get the following problem $(\widehat{P}_1)$:

$$\widehat{J}_1(x(\cdot), u(\cdot), v(\cdot)) = \int_0^\infty e^{-\rho t}\left\{\ln(1-u(t)) + \gamma\ln v(t) - \frac{\gamma v(t)}{\varkappa'\rho} + \frac{\varkappa}{\varkappa'}\ln x(t)\right\}dt \to \max,$$

$$\dot{x}(t) = \varkappa' u(t)v(t)^\gamma + (\gamma v(t) - \varkappa'\delta)x(t), \qquad x(0) = x_0, \quad (7.1)$$

$$v(t) \in (0, \infty), \qquad u(t) \in [0, 1).$$

To find optimal processes in the case of $0 < \gamma < 1$ and $0 < \varkappa < 1$, we apply the version of the Pontryagin maximum principle (PMP) from Aseev and Veliov [13] (we reformulate it in the current value form). This version of the PMP contains an additional Cauchy-type formula for the (current value) adjoint variable $p(\cdot)$, which in our case reads

$$p(t) = \frac{\varkappa e^{\rho t}}{\varkappa' y_*(t)} \int_t^\infty e^{-\rho s} \frac{y_*(s)}{x_*(s)}\, ds, \tag{7.2}$$

where $x_*(\cdot)$ is an optimal trajectory corresponding to an optimal control $(u_*(\cdot), v_*(\cdot))$ and $y_*(\cdot)$ is a solution to the linearized equation

$$\dot{y}(t) = (\gamma v_*(t) - \varkappa'\delta)y(t), \qquad y(0) = 1. \tag{7.3}$$

Theorem 2. *If $(x_*(\cdot), u_*(\cdot), v_*(\cdot))$ is an optimal admissible process in problem $(\widehat{P}_1)$, then the function $p(\cdot)$ defined by (7.2) is locally absolutely continuous and satisfies the current value core conditions of the maximum principle in the normal form, namely, the current value adjoint equation*

$$\dot{p}(t) = \rho p(t) - (\gamma v_*(t) - \varkappa'\delta)p(t) - \frac{\varkappa}{\varkappa' x_*(t)} \tag{7.4}$$

and the maximum condition

$$\mathcal{M}(x_*(t), u_*(t), v_*(t), p(t)) \overset{\text{a.e.}}{=} \sup_{u\in[0,1),\, v\in(0,\infty)} \mathcal{M}(x_*(t), u, v, p(t)), \tag{7.5}$$

where

$$\mathcal{M}(x, u, v, p) = \varkappa' u v^\gamma p + (\gamma v - \varkappa'\delta)xp + \ln(1-u) + \gamma \ln v - \frac{\gamma v}{\varkappa'\rho} + \frac{\varkappa}{\varkappa'} \ln x, \tag{7.6}$$

$x > 0$, $u \in [0, 1)$, $v \in (0, \infty)$, $p \in \mathbb{R}$, is the current value Hamilton–Pontryagin function for problem $(\widehat{P}_1)$.

Proof. The statement follows from Theorem 3.3 in Aseev and Veliov [13]. We only need to check assumptions (A1) and (A2) from Aseev and Veliov [13] for any admissible process $(x_*(\cdot), u_*(\cdot), v_*(\cdot))$ in problem $(\widehat{P}_1)$. Let

$$g(x, u, v) := \ln(1-u) + \gamma \ln v - \frac{\gamma v}{\varkappa'\rho} + \frac{\varkappa}{\varkappa'} \ln x, \qquad f(x, u, v) := \varkappa' u v^\gamma + (\gamma v - \varkappa'\delta)x,$$

$x > 0$, $u \in [0, 1)$, $v \in (0, \infty)$, be the instantaneous utility function and the function that determines the dynamics of the state variable in problem $(\widehat{P}_1)$, respectively.

In Assumption (A1) of Aseev and Veliov [13] it is required that there should exist a continuous function $\omega(t)\colon [0, \infty) \to (0, \infty)$ such that

$$\max_{\{x\colon |x - x_*(t)| \le \omega(t)\}} \bigl\{ |f_x(x, u_*(t), v_*(t))| + e^{-\rho t} |g_x(x, u_*(t), v_*(t))| \bigr\}$$
$$= \gamma v_*(t) - \varkappa'\delta + e^{-\rho t} \max_{\{x\colon |x - x_*(t)| \le \omega(t)\}} \frac{\varkappa}{\varkappa' x_*(t)}$$

is a locally integrable function of t (here, $f_x(\cdot)$ and $g_x(\cdot)$ are the partial derivatives of $f(\cdot)$ and $g(\cdot)$ with respect to x). To satisfy this assumption, it suffices to take $\omega(t) = x_*(t)/2$

for $t \geq 0$. Indeed, according to (7.1) and (4.4), $x_*(\cdot)$ is a positive continuous function, and the function $v_*(\cdot)$ is locally integrable by the definition of an admissible control in problem $(\widetilde{P}_1)$ (see Section 4).

In Assumption (A2) of Aseev and Veliov [13] it is required that there should exist a number $\beta > 0$ and an integrable function $\lambda\colon [0,\infty) \to [0,\infty)$ such that for every $\mathfrak{x}$ with $|\mathfrak{x} - x_0| < \beta$ equation (7.1) with $u(\cdot) = u_*(\cdot)$, $v(\cdot) = v_*$ and the initial condition $x(0) = \mathfrak{x}$ has a solution $x(\mathfrak{x};\cdot)$ and

$$\max_{x\in\operatorname{conv}\{x(\mathfrak{x};t),x_*(t)\}} \left|e^{-\rho t} g_x(x, u_*(t), v_*(t))(x(\mathfrak{x};t) - x_*(t))\right| \overset{\text{a.e.}}{\leq} |\mathfrak{x} - x_0|\lambda(t), \tag{7.7}$$

where $\operatorname{conv}\{x(\mathfrak{x};t), x_*(t)\}$ is the closed interval with endpoints $x(\mathfrak{x};t)$ and $x_*(t)$. Clearly, $x(\mathfrak{x};\cdot) - x_*(\cdot)$ satisfies equation (7.3) with the initial value $\mathfrak{x} - x_0$ instead of 1 at $t = 0$. So $x(\mathfrak{x};t) - x_*(t) = (\mathfrak{x} - x_0)y_*(t)$ for all $t \geq 0$. Then condition (7.7) takes the form

$$\frac{\varkappa e^{-\rho t} y_*(t)}{\varkappa' \min\{x(\mathfrak{x};t), x_*(t)\}} \leq \lambda(t).$$

Since $u_*(t)v_*(t)^\gamma \geq 0$ for all t, we have $x(\mathfrak{x};t) \geq \mathfrak{x}y_*(t)$ and $x_*(t) \geq x_0 y_*(t)$. Therefore, it suffices to take $\beta = x_0/2$ and $\lambda(t) = 2\varkappa e^{-\rho t}/(\varkappa' x_0)$ for $t \geq 0$. $\square$

Corollary 1. *If $(x_*(\cdot), u_*(\cdot), v_*(\cdot))$ is an optimal admissible process in problem $(\widehat{P}_1)$, then the current value adjoint variable $p(\cdot)$ (7.2) is positive on $[0,\infty)$ and the estimate $0 < p(t)x_*(t) \leq \varkappa/(\varkappa'\rho)$ is valid for all $t \geq 0$.*

Proof. The positivity is clear from (7.2). To show the boundedness, note that the ratio $x_*(\cdot)/y_*(\cdot)$ does not decrease, because its time derivative (equal to $\varkappa' u_*(\cdot)v_*(\cdot)^\gamma/y_*(\cdot)$) is nonnegative. Therefore,

$$p(t)x_*(t) = \frac{\varkappa e^{\rho t} x_*(t)}{\varkappa' y_*(t)} \int_t^\infty e^{-\rho s}\frac{y_*(s)}{x_*(s)}\, ds \leq \frac{\varkappa e^{\rho t}}{\varkappa'} \int_t^\infty e^{-\rho s}\, ds = \frac{\varkappa}{\varkappa'\rho}.$$

$\square$

Now we will analyze the Hamiltonian system of the PMP for problem $(\widehat{P}_1)$, which consists of equations (7.1) and

$$\dot{p}(t) = \rho p(t) - (\gamma v(t) - \varkappa'\delta)p(t) - \frac{\varkappa}{\varkappa' x(t)} \tag{7.8}$$

(see (7.4)) with $u(\cdot)$ and $v(\cdot)$ expressed in terms of $x(\cdot)$ and $p(\cdot)$ by means of the maximum condition (7.5). According to Corollary 1, it suffices to focus on the set

$$\Gamma := \left\{(x,p) \in \mathbb{R}^2\colon\ x > 0,\ p > 0,\ xp \leq \frac{\varkappa}{\varkappa'\rho}\right\}, \tag{7.9}$$

i.e., to consider only those trajectories $(x(\cdot), p(\cdot))$ that completely lie in Γ (for all $t \geq 0$).

To solve the maximum condition with respect to $u(\cdot)$ and $v(\cdot)$, it will be convenient to introduce an auxiliary control variable $\widehat{u} = (1 - u)v^\gamma$ (instead of u) subject to the constraint $0 < \widehat{u} \le v^\gamma$. In the variables $x, \widehat{u}, v, p$, the current value Hamilton–Pontryagin function (7.6) takes the form

$$\widehat{\mathcal{M}}(x,\widehat{u},v,p) = \varkappa'(v^\gamma - \widehat{u})p + (\gamma v - \varkappa'\delta)xp + \ln\widehat{u} - \frac{\gamma v}{\varkappa'\rho} + \frac{\varkappa}{\varkappa'}\ln x, \tag{7.10}$$

$x > 0$, $\widehat{u} \in (0, v^\gamma]$, $v \in (0,\infty)$, $p \in \mathbb{R}$.

Resolving the maximum condition by differentiating (7.10) with respect to either $\widehat{u}$ or v, we find

$$\widehat{u} = \frac{1}{\varkappa' p}, \qquad v = \left(\frac{\varkappa' p}{\frac{1}{\varkappa'\rho} - xp}\right)^{1/(1-\gamma)} \qquad \text{provided that} \quad \frac{1}{\varkappa'\rho} - xp \le (\varkappa' p)^{1/\gamma} \tag{7.11}$$

(recall that $xp \le \frac{\varkappa}{\varkappa'\rho} < \frac{1}{\varkappa'\rho}$ in Γ). If $\frac{1}{\varkappa'\rho} - xp > (\varkappa' p)^{1/\gamma}$, then $\widehat{u} = v^\gamma$ (i.e., $u = 0$) and so

$$\widehat{\mathcal{M}}(x,\widehat{u},v,p) = (\gamma v - \varkappa'\delta)xp + \gamma\ln v - \frac{\gamma v}{\varkappa'\rho} + \frac{\varkappa}{\varkappa'}\ln x.$$

Calculating the derivative with respect to v, we find that the maximum is achieved at

$$\widehat{u} = \frac{1}{\left(\frac{1}{\varkappa'\rho} - xp\right)^\gamma}, \qquad v = \frac{1}{\frac{1}{\varkappa'\rho} - xp} \qquad \text{provided that} \quad \frac{1}{\varkappa'\rho} - xp > (\varkappa' p)^{1/\gamma}. \tag{7.12}$$

Let

$$\begin{aligned} \Gamma_0 &= \left\{(x,p) \in \Gamma\colon\ xp < \frac{1}{\varkappa'\rho} - (\varkappa' p)^{1/\gamma}\right\}, \\ \Gamma_1 &= \left\{(x,p) \in \Gamma\colon\ xp > \frac{1}{\varkappa'\rho} - (\varkappa' p)^{1/\gamma}\right\} \end{aligned} \tag{7.13}$$

be the domains of zero and nonzero control u, respectively, and $\Gamma_{01} = \overline{\Gamma}_0 \cap \overline{\Gamma}_1 \cap \Gamma$ be the curve separating them (see Fig. 1).

Let us find stationary points of system (7.1), (7.8) in $\overline{\Gamma}_0$. At such a point we must have $u = 0$, $v = \varkappa'\delta/\gamma$ (see (7.1)), $xp = \varkappa/(\varkappa'\rho)$ (see (7.8)), so $v = \rho$ (see (7.12)), i.e., $\gamma\rho = \varkappa'\delta$, and $\rho^\gamma\varkappa' p \le 1$ (see (7.13)). Thus, stationary points exist in $\overline{\Gamma}_0$ if and only if $\gamma\rho = \varkappa'\delta$, and in this case they fill the boundary curve $\overline{\Gamma_0 \cap \partial\Gamma}$.

To facilitate the analysis of system (7.1), (7.8) in $\overline{\Gamma}_1$, we make the following change of variables:

$$(x,p) \to (\theta,\nu), \qquad \theta = \varkappa' xp, \qquad \nu = \left(\frac{\frac{1}{\varkappa'\rho} - xp}{\varkappa' p}\right)^{1/(1-\gamma)}. \tag{7.14}$$

In the new variables, the set Γ_1 is described by the inequalities

$$0 < \theta \le \frac{\varkappa}{\rho}, \qquad 0 < \varkappa'\nu < \frac{1}{\rho} - \theta, \tag{7.15}$$

and the reverse change is given by

$$x = \frac{\varkappa'\theta\nu^{1-\gamma}}{\frac{1}{\rho} - \theta}, \qquad p = \frac{\frac{1}{\rho} - \theta}{\varkappa'^2\nu^{1-\gamma}}. \tag{7.16}$$

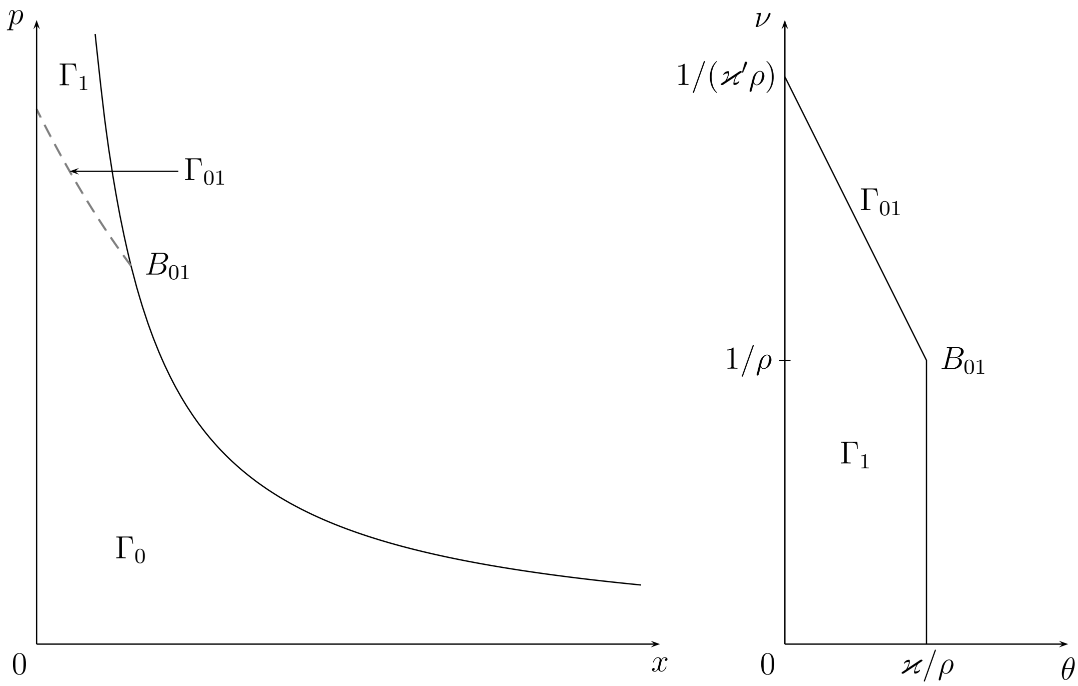

Figure 1: The sets Γ_0 and Γ_1 separated by the curve Γ_{01} (with $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_{01}$) in the coordinates (x, p), and the set Γ_1 in the coordinates (θ, ν)

Next, we write system (7.1), (7.8) in the variables (θ, ν):

$$\dot\theta(t) = \varkappa'[x(t)\dot p(t) + p(t)\dot x(t)] = \rho\theta(t) - \varkappa + \varkappa'^2\big[v^\gamma(x(t), p(t)) - \widehat{u}(x(t), p(t))\big]p(t) \tag{7.17}$$
$$= \rho\theta(t) - \varkappa + \frac{\varkappa'^2 p(t)}{\nu(t)^\gamma} - \varkappa' = \rho\theta(t) - 1 + \frac{\frac{1}{\rho} - \theta(t)}{\nu(t)} = \left(\frac{1}{\rho} - \theta(t)\right)\left(\frac{1}{\nu(t)} - \rho\right),$$

$$\dot\nu(t) = \frac{\nu(t)^\gamma}{1-\gamma}\left[-\frac{\dot\theta(t)}{\varkappa'^2 p(t)} - \nu(t)^{1-\gamma}\frac{\dot p(t)}{p(t)}\right] \tag{7.18}$$
$$= -\frac{\nu(t)}{1-\gamma}\left[\frac{\left(\frac{1}{\rho} - \theta(t)\right)\left(\frac{1}{\nu(t)} - \rho\right)}{\left(\frac{1}{\rho} - \theta(t)\right)} + \rho - \frac{\gamma}{\nu(t)} + \varkappa'\delta - \frac{\varkappa}{\theta(t)}\right]$$
$$= -\frac{\nu(t)}{1-\gamma}\left[\frac{1-\gamma}{\nu(t)} + \varkappa'\delta - \frac{\varkappa}{\theta(t)}\right] = -1 - \frac{\varkappa'\delta}{1-\gamma}\nu(t) + \frac{\varkappa}{1-\gamma}\frac{\nu(t)}{\theta(t)}.$$

Thus, in Γ_1 we have

$$\begin{cases} \dot\theta(t) = \left(\dfrac{1}{\rho} - \theta(t)\right)\left(\dfrac{1}{\nu(t)} - \rho\right), \\ \dot\nu(t) = -1 - \dfrac{\varkappa'\delta}{1-\gamma}\nu(t) + \dfrac{\varkappa}{1-\gamma}\dfrac{\nu(t)}{\theta(t)}. \end{cases} \tag{7.19}$$

Since $\theta(t) < 1/\rho$ for all t, at a stationary point we must have

$$\nu = \frac{1}{\rho}, \qquad \theta = \frac{\varkappa\nu}{1-\gamma+\varkappa'\delta\nu} = \frac{\varkappa}{(1-\gamma)\rho + \varkappa'\delta}, \tag{7.20}$$

which corresponds (see (7.16)) to

$$x = x_1 := \frac{\varkappa\varkappa' \rho^\gamma}{(\varkappa' - \gamma)\rho + \varkappa'\delta}, \qquad p = p_1 := \frac{(\varkappa' - \gamma)\rho + \varkappa'\delta}{\varkappa'^2 \rho^\gamma((1-\gamma)\rho + \varkappa'\delta)}. \tag{7.21}$$

This point lies in Γ_1 (see (7.15)) only if $\varkappa'\delta > \gamma\rho$ (we denote it by $B_1 = (x_1, p_1)$ in this case). If $\varkappa'\delta = \gamma\rho$, this point is the point B_{01} of intersection of the curve Γ_{01} with the curvilinear boundary of Γ (it was already found above among the stationary points in $\overline{\Gamma}_0$).

Let us show that the stationary point B_1 (7.21) is of saddle type. This statement can also be checked in the coordinates (θ, ν). According to the Grobman–Hartman theorem (Hartman [32]), to this end it suffices to show that the determinant of the Jacobian matrix of system (7.19) at the point (7.20) is negative. This determinant is equal to

$$0 \cdot \left(-\frac{\varkappa'\delta}{1-\gamma} + \frac{\varkappa}{(1-\gamma)\theta} \right) - \left(\frac{1}{\rho} - \theta \right) \frac{1}{\nu^2} \frac{\varkappa\nu}{(1-\gamma)\theta^2} < 0.$$

The next important point of our analysis is to determine the direction in which the trajectories of system (7.1), (7.8) cross the curve Γ_{01}. In the coordinates (θ, ν), Γ_{01} is given by the equation $\varkappa'\nu + \theta = 1/\rho$. Let us calculate the scalar product of the gradient of the left-hand side of this equation (considered as a function of θ and ν) and the right-hand side of system (7.19):

$$\begin{aligned}\left(\frac{1}{\rho} - \theta \right) \left(\frac{1}{\nu} - \rho \right) + \varkappa' \left(-1 - \frac{\varkappa'\delta}{1-\gamma}\nu + \frac{\varkappa}{1-\gamma}\frac{\nu}{\theta} \right) &= \varkappa' - \varkappa'\rho\nu - \varkappa' - \frac{\varkappa'\nu}{1-\gamma}\left(\varkappa'\delta - \frac{\varkappa}{\theta} \right) \\ &= -\frac{\varkappa'\nu}{1-\gamma}\left((1-\gamma)\rho + \varkappa'\delta - \frac{\varkappa}{\theta} \right).\end{aligned}$$

So, for $\theta < \varkappa/((1-\gamma)\rho + \varkappa'\delta)$ the trajectories of our system cross Γ_{01} from Γ_1 to Γ_0, while for $\theta > \varkappa/((1-\gamma)\rho + \varkappa'\delta)$ the trajectories cross Γ_{01} from Γ_0 to Γ_1.

Note also that (see (7.17), (7.12))

$$\dot{\theta}(t) = \rho\theta(t) - \varkappa \qquad \text{in } \Gamma_0. \tag{7.22}$$

Now we are ready to describe the qualitative behavior of an optimal process in problem $(\widehat{P}_1)$. We will see that for any initial condition $x_0 > 0$, system (7.1), (7.8) has only one trajectory with $x(0) = 0$ that completely lies in Γ. Since problem $(\widehat{P}_1)$ has an optimal solution (by Theorem 1) and the trajectory of system (7.1), (7.8) that corresponds to this solution must lie in Γ (by Theorem 2), the unique trajectory lying in Γ will correspond to a (unique) optimal solution. For brevity, we will call this trajectory an optimal trajectory.

(a) Let first $\varkappa'\delta < \gamma\rho$, i.e., $\varkappa + (\rho/\delta)\gamma > 1$. In this case, there are no stationary points in Γ. Hence, the x-coordinate must vary monotonically along an optimal solution (Theorem 4.4 in Aseev and Kryazhimskiy [10]). Next, we have $\theta \le \varkappa/\rho < \varkappa/((1-\gamma)\rho + \varkappa'\delta)$ on Γ_{01} (and even everywhere in Γ), so a trajectory can only leave Γ_1 and enter Γ_0, but cannot reach Γ_1 from Γ_0. In addition, the set Γ_1 on the (θ, ν)-plane is bounded, so any trajectory that starts in Γ_1 leaves Γ_1 in finite time. Therefore, the tail of an optimal trajectory (or the whole such trajectory) must lie in Γ_0. In Γ_0 we have $u = 0$, so

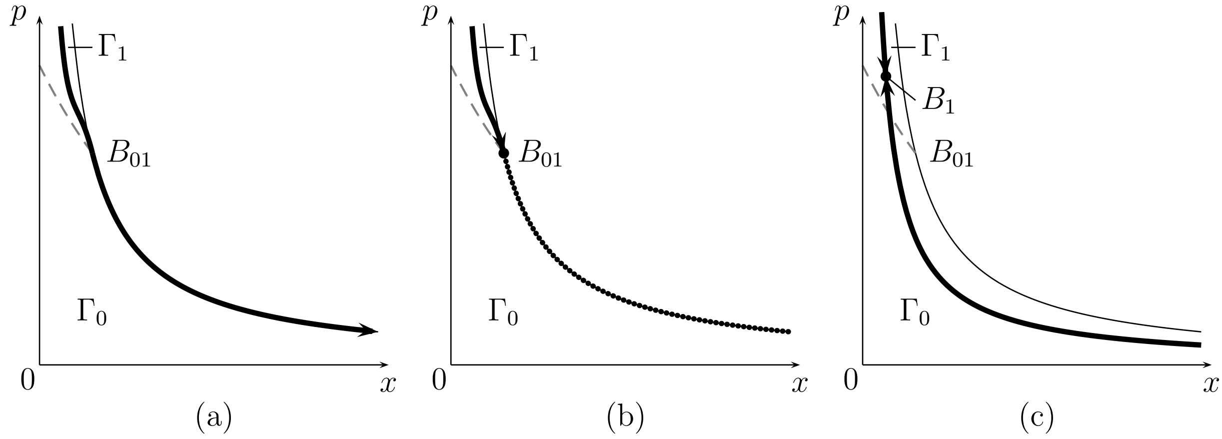

Figure 2: Optimal trajectories (shown by thick lines): (a) for $\varkappa'\delta < \gamma\rho$; (b) for $\varkappa'\delta = \gamma\rho$; and (c) for $\varkappa'\delta > \gamma\rho$. All other trajectories leave Γ in finite time

$y(t) = \mathrm{const}\cdot x(t)$ (see (7.1), (7.3)), and hence $p(\cdot)x(\cdot) \equiv \varkappa/(\varkappa'\rho)$ (see (7.2) or (7.22)) and $v(\cdot) \equiv \rho$ (see (7.12)) for an optimal trajectory.

Let $x_{01} = \varkappa/\rho^{1-\gamma}$ be the x-coordinate of the point B_{01} of intersection of Γ_{01} with the curve $xp = \varkappa/(\varkappa'\rho)$ (see Fig. 1). It follows from the above analysis that if $x_0 \geq x_{01}$, then the optimal controls are $u_*(\cdot) \equiv 0$, $v_*(\cdot) \equiv \rho$, the optimal trajectory on the (x,p)-plane coincides with the curve $xp = \varkappa/(\varkappa'\rho)$, $x \geq x_0$, and $x_*(t) = x_0 e^{(\gamma\rho - \varkappa'\delta)t}$, $t \geq 0$. If $x_0 < x_{01}$, then the optimal trajectory starts in Γ_1, reaches the point $B_{01} \in \Gamma_{01}$ in finite time, and then follows the curve $xp = \varkappa/(\varkappa'\rho)$, $x \geq x_0$, as described above (see Fig. 2a). Obviously, there is only one such trajectory (i.e., there is only one integral curve in Γ_1 that passes through B_{01}, because B_{01} is a regular point for the Hamiltonian system of the PMP in Γ_1). Explicit formulas (if any) for the Γ_1-part of the optimal trajectory are hard to find in this case (and in all the other cases as well).

(b) Now let $\varkappa'\delta = \gamma\rho$, i.e., $\varkappa + (\rho/\delta)\gamma = 1$. Then B_{01} and all points on the curve $xp = \varkappa/(\varkappa'\rho)$ to the right of B_{01} (with $x \geq x_{01}$) are stationary points of system (7.1), (7.8). For the same reasons as above, a trajectory can leave Γ_1 and enter Γ_0, but cannot reach Γ_1 from Γ_0. However, there is also a trajectory that lies in Γ_1 and tends to the point B_{01}. If $x_0 \geq x_{01}$, then the optimal controls are again $u_*(\cdot) \equiv 0$, $v_*(\cdot) \equiv \rho$, the optimal trajectory on the (x,p)-plane coincides with the point $(x_0, \varkappa/(\varkappa'\rho x_0))$, and $x_*(\cdot) \equiv x_0$. If $x_0 < x_{01}$, then the optimal trajectory must lie in Γ_1 and tend to the point B_{01}. (Obviously, such a trajectory cannot reach Γ_0 at a point different from B_{01}.) The existence of at least one such trajectory follows from Theorem 1. It remains to show that for each $x_0 < x_{01}$ there is only one such trajectory (i.e., there is only one trajectory in Γ_1 that tends to B_{01}; see Fig. 2b).

Indeed, since B_{01} is a saddle point (see above) for any smooth extension of system (7.1), (7.8) to a neighborhood of B_{01}, there may be at most two trajectories with $x(0) = x_0$ that tend to B_{01} from Γ_1. In the coordinates (θ, ν), $B_{01} = (\varkappa/\rho, 1/\rho)$. Note that $\dot\theta(t) > 0$ if $\nu(t) < 1/\rho$, $\dot\theta(t) < 0$ if $\nu(t) > 1/\rho$, and

$$\dot\nu(t) = \frac{1}{\rho(1-\gamma)}\left(\frac{\varkappa}{\theta(t)} - \rho(1-\gamma) - \varkappa'\delta\right) = \frac{1}{\rho(1-\gamma)}\left(\frac{\varkappa}{\theta(t)} - \rho\right) \geq 0 \qquad \text{for} \quad \nu(t) = \frac{1}{\rho}.$$

So, if there were two trajectories that tend to B_{01} from Γ_1, then we would have $\nu(t) < 1/\rho$ for both of them and hence all trajectories that lie between these two trajectories would also tend to B_{01} (because $\dot{\theta}(t) > 0$ on all such trajectories). However, this is impossible for a saddle point.

(c) Finally, let $\varkappa'\delta > \gamma\rho$, i.e., $\varkappa + (\rho/\delta)\gamma < 1$. In this case, system (7.1), (7.8) has only one stationary point B_1 (7.21), which belongs to Γ_1. Again, as explained above, the x-coordinate must vary monotonically along an optimal solution. Moreover, in Γ_0 we have

$$\dot{x}(t) = (\gamma v(t) - \varkappa'\delta)x(t) \le (\gamma\rho - \varkappa'\delta)x(t) < 0.$$

Similarly, in Γ_1 we have $\dot{p}(t) < 0$ for large $p(t)$. So any trajectory that completely lies in Γ must tend to B_1. Since it is a saddle point, there are exactly two integral curves of system (7.1), (7.8) along which trajectories tend to B_1 (see Fig. 2c).

Let us summarize the above analysis for $0 < \gamma < 1$ and $0 < \varkappa < 1$.

Theorem 3. *Let $0 < \gamma < 1$ and $0 < \varkappa < 1$. For any initial state $x_0 > 0$ in problem $(\widehat{P}_1)$ (or $(\widetilde{P}_1)$), there is only one trajectory of the Hamiltonian system of the PMP with $x(0) = x_0$ that satisfies the necessary optimality conditions formulated in Theorem 2. Therefore, this trajectory corresponds to a unique optimal solution in problem $(\widehat{P}_1)$ (or $(\widetilde{P}_1)$). The x-coordinate varies monotonically along this trajectory, and so the p-coordinate can be expressed as a well-defined function of x: $p = p(x)$, $x > 0$. The optimal controls can then be represented in the form of optimal synthesis, $u_* = u_*(x)$ and $v_* = v_*(x)$, via formulas (7.11) for $x \le x_{01} = \varkappa/\rho^{1-\gamma}$ and (7.12) for $x > x_{01}$, with $p = p(x)$.*

The qualitative behavior of the optimal trajectory of the Hamiltonian system of the PMP is shown in Fig. 2.

If $\varkappa + (\rho/\delta)\gamma > 1$, then $x_*(t) \to \infty$ as $t \to \infty$, with $u_*(x) = 0$ and $v_*(x) = \rho$ for $x \ge x_{01}$.

If $\varkappa + (\rho/\delta)\gamma = 1$, then $x_*(t) \to x_{01}$, $u_*(t) \to 0$ and $v_*(t) \to \rho$ as $t \to \infty$ for $x_0 < x_{01}$, and $x_*(t) \equiv x_0$, $u_*(t) \equiv 0$ and $v_*(t) \equiv \rho$ for $x_0 \ge x_{01}$.

If $\varkappa + (\rho/\delta)\gamma < 1$, then $x_*(t) \to x_1$, $u_*(t) \to 1 - 1/((1-\varkappa)p_1\rho^\gamma)$ and $v_*(t) \to \rho$ as $t \to \infty$, with x_1 and p_1 given by (7.21).

In all cases, $x_*(\cdot)$ is a monotonic function of t.

7.2 Weak scale effects in resource use ($0 < \gamma < 1$) and strong scale effects in capital accumulation ($\varkappa = 1$)

Since $0 < \gamma < 1$, Theorem 1(i) guarantees the existence of an optimal process in problem $(\widetilde{P}_1)$. For $\varkappa = 1$, retaining the term $\ln R(t)$ in the objective functional (3.4) and transforming $\ln K(t)$ as in (4.6), we come to the objective functional

$$\widehat{J}_1(u(\cdot), R(\cdot)) = \int_0^\infty e^{-\rho t}\left[\ln(1 - u(t)) + \frac{u(t)R(t)^\gamma}{\rho} + \gamma \ln R(t)\right] dt,$$

which coincides with (3.4) up to a constant term. The only constraints imposed on the controls here are given by (3.6). In an optimal regime we must obviously have

$$u_*(t) = \begin{cases} 0 & \text{if } R(t)^\gamma < \rho, \\ 1 - \rho/R(t)^\gamma & \text{if } R(t)^\gamma \geq \rho, \end{cases} \qquad t \geq 0,$$

so

$$\widehat{J}_1(u_*(\cdot), R(\cdot)) = \int_{R(t)^\gamma \geq \rho} e^{-\rho t}\left[\ln\rho + \frac{R(t)^\gamma}{\rho} - 1\right] dt + \int_{R(t)^\gamma < \rho} e^{-\rho t}\gamma \ln R(t)\, dt. \tag{7.23}$$

Since the function

$$\mathcal{F}(r) = \begin{cases} \gamma \ln r, & 0 < r^\gamma < \rho, \\ \ln\rho + r^\gamma/\rho - 1, & r^\gamma \geq \rho, \end{cases}$$

is monotonically increasing, it is clear that replacing any control $R(\cdot)$ with its nonincreasing rearrangement does not decrease the value of the objective functional (and increases it if $R(\cdot)$ is not an "essentially" monotonically nonincreasing function). So all optimal controls $R(\cdot)$ are contained in the class of monotonically nonincreasing functions, and we assume below in this section that $R(\cdot)$ is such a function. Then we can rewrite (7.23) as

$$\widehat{J}_1(u_*(\cdot), R(\cdot)) = \int_0^T e^{-\rho t}\left[\ln\rho + \frac{R(t)^\gamma}{\rho} - 1\right] dt + \int_T^\infty e^{-\rho t}\gamma \ln R(t)\, dt, \tag{7.24}$$

where $T = T(R(\cdot)) = \inf\{t \geq 0\colon R(t)^\gamma < \rho\}$.

Let us fix $T \geq 0$ and $S(T)$ for a while, with $0 < S(T) \leq S_0 - \rho^{1/\gamma}T$ (recall that $R(t) \geq \rho^{1/\gamma}$ on $[0,T)$) and hence $T < S_0\rho^{-1/\gamma}$. Then the problem of maximizing $\widehat{J}_1(u_*(\cdot), R(\cdot))$ splits into two independent maximization problems: one on $[0,T]$, and the other on $[T,\infty)$.

The problem of maximizing the last integral on the right-hand side of (7.24) under the constraints $\int_T^\infty R(t)\, dt \leq S(T)$ and $R(t) \leq \rho^{1/\gamma}$, $t > T$, can be solved by integrating by parts as in (4.9) and then maximizing the integrand at every point. This yields $v_*(t) \equiv \rho$, $t > T$, and so $S_*(t) = S(T)e^{-\rho(t-T)}$, $R_*(t) = \rho S(T)e^{-\rho(t-T)}$, $t > T$, and

$$\int_T^\infty e^{-\rho t}\gamma \ln R_*(t)\, dt = \gamma e^{-\rho T}\left[\frac{\ln S(T)}{\rho} + \frac{\ln\rho - 1}{\rho}\right] \tag{7.25}$$

(see (4.9)) provided that $S(T) \leq \rho^{(1-\gamma)/\gamma}$. (If $S(T) > \rho^{(1-\gamma)/\gamma}$, then we would have $R_*(t) = \rho^{1/\gamma}$ for $T < t < T_1 := T + S(T)\rho^{-1/\gamma} - \rho^{-1}$, but this solution corresponds to another pair $(T, S(T))$, namely, to a pair $(T_1, \rho^{(1-\gamma)/\gamma})$. Therefore, we can assume without loss of generality that the condition $S(T) \leq \rho^{(1-\gamma)/\gamma}$ always holds.)

By Theorem 1(i), an optimal admissible process $(R_*(\cdot), S_*(\cdot))$ is admissible in problem $(\widetilde{P}_1^2)$; i.e., $R_*(t) = v_*(t)S_*(t) \leq V_2(t)S_0$ for $t \geq 0$. This optimal process must also be optimal in the problem on any fixed time interval $[0,T']$ for the state variable $S(\cdot)$, $\dot{S}(t) = -R(t)$, with fixed initial and terminal values S_0 and $S_*(T')$, respectively, and with the objective functional $\int_0^{T'} e^{-\rho t}\mathcal{F}(R(t))\, dt \to \max$. Since $R_*(\cdot)$ is bounded on $[0,T']$, we can apply the PMP (Pontryagin et al. [42]) to this finite horizon problem. According to the PMP, there exists an absolutely continuous function $\psi(\cdot)$ and a constant $\psi_0 \geq 0$ such that, first,

$$\dot\psi(t) \equiv 0 \qquad \text{for a.e.} \quad t \in [0, T'],$$

i.e., $\psi(\cdot)$ is a constant function, say $\psi(\cdot) \equiv c$, and, second,

$$-cR_*(t) + \psi_0 e^{-\rho t}\mathcal{F}(R_*(t)) = \max_{0<R<R_{\max}} \bigl[-cR + \psi_0 e^{-\rho t}\mathcal{F}(R)\bigr], \qquad 0 \leq t \leq T', \tag{7.26}$$

with $\psi_0 + |c| > 0$. Here $R_{\max} = R_{\max}(T')$ is a number such that certainly $R_*(t) < R_{\max}$ for $t \in [0, T']$ (we can take $R_{\max} = V_2(T')S_0 + 1$).

The maximum condition (7.26) implies that neither c nor ψ_0 is zero, so we can set $\psi_0 = 1$. Hence $R_*(t)$ is determined for a.e. $t \in [0, T']$ by the condition $\mathcal{F}'(R_*(t)) = ce^{\rho t}$, which reads

$$\frac{\gamma}{R_*(t)} = ce^{\rho t} \quad \text{for} \quad 0 < R_*(t)^\gamma < \rho, \qquad \frac{\gamma}{\rho R_*(t)^{1-\gamma}} = ce^{\rho t} \quad \text{for} \quad R_*(t)^\gamma \geq \rho,$$

or

$$R_*(t) = \begin{cases} \gamma c^{-1} e^{-\rho t}, & 0 < R_*(t)^\gamma < \rho, \\ (\gamma c^{-1}\rho^{-1})^{1/(1-\gamma)} e^{-\rho t/(1-\gamma)}, & R_*(t)^\gamma \geq \rho. \end{cases} \tag{7.27}$$

If $T = 0$ in (7.24), then, as shown above, $R_*(t) = \rho S_0 e^{-\rho t}$, $t \geq 0$. If $T > 0$, then $R_*(\cdot)$, being a solution to the equation $\mathcal{F}'(R_*(t)) = ce^{\rho t}$, is also a continuous function of t, so at the "switching" point T we must have

$$R_*(T) = \rho^{1/\gamma} = \gamma c^{-1} e^{-\rho T} = (\gamma c^{-1}\rho^{-1})^{1/(1-\gamma)} e^{-\rho T/(1-\gamma)}.$$

Solving this system, we find c and substitute it into (7.27):

$$R_*(t) = \begin{cases} \rho^{1/\gamma} e^{-\rho(t-T)}, & t > T, \\ \rho^{1/\gamma} e^{-\rho(t-T)/(1-\gamma)}, & 0 \leq t \leq T. \end{cases} \tag{7.28}$$

To find T, note that in the optimal regime the integral of $R_*(\cdot)$ must be exactly equal to S_0. Then

$$S_0 = \rho^{(1-\gamma)/\gamma} + (1-\gamma)\rho^{(1-\gamma)/\gamma}[e^{\rho T/(1-\gamma)} - 1] = \rho^{(1-\gamma)/\gamma}\bigl[\gamma + (1-\gamma)e^{\rho T/(1-\gamma)}\bigr],$$

or

$$e^{\rho T/(1-\gamma)} = \frac{S_0\rho^{-(1-\gamma)/\gamma} - \gamma}{1-\gamma} \qquad \Leftrightarrow \qquad T = \frac{1-\gamma}{\rho} \ln \frac{S_0\rho^{-(1-\gamma)/\gamma} - \gamma}{1-\gamma}. \tag{7.29}$$

Now we can summarize the above calculations for $0 < \gamma < 1$ and $\varkappa = 1$.

Theorem 4. *Let $0 < \gamma < 1$ and $\varkappa = 1$. If $S_0 \leq \rho^{(1-\gamma)/\gamma}$, then the optimal control in problem (P_1) is $R_*(t) = \rho S_0 e^{-\rho t}$, $u_*(t) = 0$ for $t \geq 0$, $S_*(t) = S_0 e^{-\rho t}$ for $t \geq 0$, and $K_*(t) = K_0 e^{-\delta t}$ for $t \geq 0$.*

If $S_0 > \rho^{(1-\gamma)/\gamma}$, then the optimal control $R_*(\cdot)$ is given by (7.28) with T determined by (7.29), $u_*(t) = 1 - e^{\gamma\rho(t-T)/(1-\gamma)}$ for $t < T$ and $u_*(t) = 0$ for $t > T$,

$$S_*(t) = \bigl(S_0 - \gamma\rho^{(1-\gamma)/\gamma}\bigr)e^{-\rho t/(1-\gamma)} + \gamma\rho^{(1-\gamma)/\gamma} = (1-\gamma)\rho^{(1-\gamma)/\gamma}e^{\rho(T-t)/(1-\gamma)} + \gamma\rho^{(1-\gamma)/\gamma}$$

for $t \le T$ and $S_*(t) = \rho^{(1-\gamma)/\gamma} e^{-\rho(t-T)}$ for $t > T$. The formula for $K_*(t)$ can be derived from the equation $\ln K_*(t) = \ln K_0 + \int_0^t [u_*(s) R_*(s)^\gamma - \delta]\, ds$.

In the form of optimal synthesis, the optimal control can be expressed as follows: $R_*(t) = (\rho S_*(t) - \gamma \rho^{1/\gamma})/(1-\gamma)$ and $u_*(t) = 1 - \rho / R_*(t)^\gamma$ when $S_*(t) > \rho^{(1-\gamma)/\gamma}$, and $R_*(t) = \rho S_*(t)$ with $u_*(t) = 0$ when $S_*(t) \le \rho^{(1-\gamma)/\gamma}$.

In the next section we consider the special knife-edge case $\gamma = 1$.

8 The knife-edge case ($\gamma = 1$)

Here, as in the previous section, we split our analysis into two subcases with $0 < \varkappa < 1$ and $\varkappa = 1$, respectively.

8.1 The knife-edge case ($\gamma = 1$) with weak scale effects in capital accumulation ($0 < \varkappa < 1$)

To begin with, note that all arguments presented in Section 7.1 up to formula (7.10) remain valid for $\gamma = 1$ as well (except that the issue of the existence of an optimal solution for $\gamma = 1$ is still to be settled). In particular, here we will deal with problem $(\widehat{P}_1)$ (see (7.1), where one should now set $\gamma = 1$) and apply Theorem 2 and Corollary 1.

For $\gamma = 1$, the current value Hamilton–Pontryagin function (7.10) takes the form

$$\widehat{\mathcal{M}}(x, \widehat{u}, v, p) = \varkappa'(v - \widehat{u})p + (v - \varkappa'\delta)xp + \ln \widehat{u} - \frac{v}{\varkappa'\rho} + \frac{\varkappa}{\varkappa'} \ln x, \tag{8.1}$$

$x > 0$, $0 < \widehat{u} \le v < \infty$, $p \in \mathbb{R}$.

Resolving the maximum condition (7.5), we find that $v = \widehat{u}$ (i.e., $u = 0$) if the point (x, p) belongs to the set

$$\Gamma_0 = \Big\{ (x, p) \in \Gamma\colon\ \varkappa' p + xp < \frac{1}{\varkappa'\rho} \Big\}.$$

Then

$$\widehat{\mathcal{M}}(x, \widehat{u}, v, p) = (\widehat{u} - \varkappa'\delta)xp + \ln \widehat{u} - \frac{\widehat{u}}{\varkappa'\rho} + \frac{\varkappa}{\varkappa'} \ln x,$$

and so

$$\widehat{u} = v = \frac{1}{\frac{1}{\varkappa'\rho} - xp} \qquad \text{for} \quad (x, p) \in \Gamma_0. \tag{8.2}$$

Obviously,

$$\widehat{u} = \frac{1}{\varkappa' p} = \frac{1}{\frac{1}{\varkappa'\rho} - xp} \qquad \text{and} \qquad v \in [\widehat{u}, \infty) \text{ is arbitrary} \qquad \text{for} \quad (x, p) \in \Gamma_{01}, \tag{8.3}$$

where

$$\Gamma_{01} = \Big\{ (x, p) \in \Gamma\colon\ \varkappa' p + xp = \frac{1}{\varkappa'\rho} \Big\}.$$

At the other points of Γ (i.e., in the set Γ_1 defined by (7.13) with $\gamma = 1$), the maximum condition has no solution, because $\widehat{\mathcal{M}}(\cdot)$ indefinitely increases as $v \to \infty$.

As in Section 7.1, to facilitate the further analysis, we make the following change of variables (analogous to (7.14)):

$$(x,p) \to (\theta,\nu), \qquad \theta = \varkappa' x p, \qquad \nu = \frac{\frac{1}{\varkappa'\rho} - xp}{\varkappa' p}.$$

In the new variables, the set Γ_0 is described by the inequalities

$$0 < \theta \le \frac{\varkappa}{\rho}, \qquad 1 < \nu < \infty,$$

and the reverse change is given by

$$x = \frac{\varkappa'\theta\nu}{\frac{1}{\rho} - \theta}, \qquad p = \frac{\frac{1}{\rho} - \theta}{\varkappa'^2\nu}. \tag{8.4}$$

Next, we write system (7.1), (7.8) in $\overline{\Gamma}_0$ in the variables (θ,ν). Let $\widehat{v} := v - \widehat{u} = vu$; thus $\widehat{v} = 0$ in Γ_0 and $\widehat{v}$ is an arbitrary nonnegative number on Γ_{01}. Then we have

$$\dot\theta(t) = \varkappa'[x(t)\dot p(t) + p(t)\dot x(t)] = \rho\theta(t) - \varkappa + \varkappa'^2\widehat{v}p(t) = \rho\theta(t) - \varkappa + \widehat{v}\frac{\frac{1}{\rho} - \theta(t)}{\nu(t)},$$

$$\begin{aligned}
\dot\nu(t) &= -\frac{\dot\theta(t)}{\varkappa'^2 p(t)} - \nu(t)\frac{\dot p(t)}{p(t)} = -\frac{\nu(t)\dot\theta(t)}{\frac{1}{\rho} - \theta(t)} - \nu(t)\left[\rho - \frac{\varkappa'}{\frac{1}{\rho} - \theta(t)} - \widehat{v} + \varkappa'\delta - \frac{\varkappa}{\theta(t)}\right] \\
&= \frac{\nu(t)(\varkappa - \rho\theta(t))}{\frac{1}{\rho} - \theta(t)} - \widehat{v} - (\rho + \varkappa'\delta - \widehat{v})\nu(t) + \frac{\varkappa'\nu(t)}{\frac{1}{\rho} - \theta(t)} + \frac{\varkappa\nu(t)}{\theta(t)} \\
&= \left(\frac{\varkappa}{\theta(t)} - \varkappa'\delta\right)\nu(t) + \widehat{v}(\nu(t) - 1).
\end{aligned}$$

In particular, we have

$$\begin{cases} \dot\theta(t) = \rho\theta(t) - \varkappa, \\ \dot\nu(t) = \left(\dfrac{\varkappa}{\theta(t)} - \varkappa'\delta\right)\nu(t) \end{cases} \quad \text{in } \Gamma_0 = \left\{(\theta,\nu)\colon\ 0 < \theta \le \frac{\varkappa}{\rho},\ 1 < \nu < \infty\right\} \tag{8.5}$$

(note that equation (7.22) for $\dot\theta(\cdot)$ is preserved) and

$$\begin{cases} \dot\theta(t) = \rho\theta(t) - \varkappa + \widehat{v}\left(\dfrac{1}{\rho} - \theta(t)\right), \\ \dot\nu(t) = \dfrac{\varkappa}{\theta(t)} - \varkappa'\delta \end{cases} \quad \text{on } \Gamma_{01} = \left\{(\theta,\nu)\colon\ 0 < \theta \le \frac{\varkappa}{\rho},\ \nu = 1\right\}. \tag{8.6}$$

It is now easy to see that there are no stationary points in $\overline{\Gamma}_0$ if $\varkappa'\delta < \rho$, stationary points fill the boundary line $\theta = \varkappa/\rho$ if $\varkappa'\delta = \rho$, and there is only one stationary point

$\theta = \varkappa/(\varkappa'\delta)$, $\nu = 1$ (which lies on Γ_{01} and corresponds to $\widehat{v} = \varkappa\rho(\varkappa'\delta-\rho)/(\varkappa'\delta-\varkappa\rho) > 0$) if $\varkappa'\delta > \rho$. So we again have three cases, as in Section 7.1.

(a) Let first $\varkappa'\delta < \rho$, i.e., $\varkappa + \rho/\delta > 1$. Then we have $\dot{\nu}(t) > 0$ for all t along any trajectory, and if $x_0 \geq x_{01} = \varkappa$, the only trajectory that completely lies in $\overline{\Gamma}_0$ follows the curve $\theta = \varkappa/\rho$, i.e., $px = \varkappa/(\varkappa'\rho)$ (as in the Γ_0-part of Fig. 2a). According to (8.2), we have $u_*(\cdot) \equiv 0$, $v_*(\cdot) \equiv \rho$ and $x_*(t) = x_0 e^{(\rho-\varkappa'\delta)t}$ for $t \geq 0$. Below, we will show that this solution is indeed optimal (and thus complete the proof of Theorem 1 in the case when $\gamma = 1$ and $0 < \varkappa < 1$).

If $x_0 < \varkappa$, then all trajectories with $x(0) = x_0$ leave $\overline{\Gamma}_0$ in finite time, i.e., no trajectory satisfies the necessary optimality conditions, and so the problem has no optimal solution.

(b) Now let $\varkappa'\delta = \rho$, i.e., $\varkappa+\rho/\delta = 1$. Then all points on the curve $xp = \varkappa/(\varkappa'\rho)$ with $x \geq \varkappa$ are stationary points of system (7.1), (7.8). If $x_0 \geq \varkappa$, then the only trajectory with $x(0) = x_0$ that completely lies in $\overline{\Gamma}_0$ is the stationary point $(x_0, \varkappa/(\varkappa'\rho x_0))$, and the corresponding controls are again $u_*(\cdot) \equiv 0$, $v_*(\cdot) \equiv \rho$ (as in the Γ_0-part of Fig. 2b). Below, we will show that this solution is indeed optimal.

If $x_0 < \varkappa$, then all trajectories with $x(0) = x_0$ leave $\overline{\Gamma}_0$ in finite time, i.e., the problem has no optimal solution.

(c) Finally, let $\varkappa'\delta > \rho$, i.e., $\varkappa + \rho/\delta < 1$. In this case, system (7.1), (7.8) has only one stationary point B with

$$\underline{x} = \frac{\varkappa\varkappa'\rho}{\varkappa'\delta - \varkappa\rho} < \varkappa, \qquad \underline{p} = \frac{\varkappa'\delta - \varkappa\rho}{\varkappa'^3\delta\rho} \tag{8.7}$$

(see (8.4) for $\theta = \varkappa/(\varkappa'\delta)$ and $\nu = 1$), which belongs to Γ_{01}. The motion along Γ_{01} is impossible (except for staying at B), because $\nu = 1$ on Γ_{01} but $\dot{\nu}(t) \neq 0$ on $\Gamma_{01} \setminus \{B\}$.

In Γ_0, we have

$$\dot{x}(t) = (v(t) - \varkappa'\delta)x(t) \leq (\rho - \varkappa'\delta)x(t) < 0, \tag{8.8}$$

so only those trajectories that reach B and then stay at B completely lie in $\overline{\Gamma}_0$. All other trajectories leave $\overline{\Gamma}_0$ in finite time (either they cross Γ_{01} or $\theta(\cdot)$ vanishes in finite time, see (8.5)). In particular, if $x_0 < \underline{x}$, then the problem has no optimal solution.

Note also that, according to (8.5), in Γ_0 we have

$$\frac{d}{dt}[\theta(t)\nu(t)] = (\rho - \varkappa'\delta)\theta(t)\nu(t).$$

Now we show that the trajectories that completely lie in $\overline{\Gamma}_0$ in cases (a)–(c) (recall that for each $x_0 > 0$ there is at most one such trajectory) correspond to optimal processes in problem $(\widehat{P}_1)$.

Proof of Theorem 1(ii) *in the case of* $\gamma = 1$ *and* $\varkappa < 1$. For $\varkappa < 1$ the nonexistence assertions of Theorem 1 follow from the analysis above, because in the corresponding cases there are no trajectories of the Hamiltonian system of the PMP for problem $(\widehat{P}_1)$ that satisfy the necessary optimality conditions (Theorem 2). So it remains to establish the existence assertions in the case of $\gamma = 1$ and $\varkappa < 1$ (for $x_0 \geq \varkappa$ in cases (a) and (b), and for $x_0 \geq \underline{x}$ in case (c)). To this end, we introduce an auxiliary problem $(\widehat{P}_1^V)$ that differs from problem $(\widehat{P}_1)$ by the presence of the additional constraint $v(t) \le V$ for all $t \ge 0$, where $V > \rho$ is a sufficiently large real number.

The existence of an optimal solution in problem $(\widehat{P}_1^V)$ (for every fixed $V > 0$) follows from Theorem 1 in Besov [20] or from Balder [15]. Exact analogs of Theorem 2 and Corollary 1 are also obviously valid for problem $(\widehat{P}_1^V)$, so an optimal trajectory must completely lie in Γ (7.9).

The maximum condition in Γ_0 is solved in exactly the same way as above (see (8.2)), because $1/(\frac{1}{\varkappa'\rho} - xp) \le \rho$ in Γ. The maximum condition on Γ_{01} now gives

$$\widehat{u} = \frac{1}{\varkappa' p} = \frac{1}{\frac{1}{\varkappa'\rho} - xp} \le \rho \qquad \text{and} \qquad v \in [\widehat{u}, V] \text{ is arbitrary} \qquad \text{for} \quad (x,p) \in \Gamma_{01}$$

(compare (8.3)). The main difference from the above analysis is that now the maximum condition is also solvable in the set Γ_1 (which is defined by (7.13) with $\gamma = 1$):

$$\widehat{u} = \frac{1}{\varkappa' p} < \frac{1}{\frac{1}{\varkappa'\rho} - xp} \le \rho \qquad \text{and} \qquad v = V \qquad \text{for} \quad (x,p) \in \Gamma_1.$$

Note that we have $x < \varkappa$ for all points $(x,p) \in \Gamma_1$. So, in cases (a) and (b), for $x_0 \ge \varkappa$ there is only one trajectory (of the Hamiltonian system of the PMP for problem $(\widehat{P}_1^V)$) with $x(0) = x_0$ that completely lies in Γ. This trajectory corresponds to a unique optimal solution in problem $(\widehat{P}_1^V)$, does not depend on V for $V > \rho$ and is the same as described above for problem $(\widehat{P}_1)$. We see that the optimal value of the functional $\widehat{J}_1(x(\cdot), u(\cdot), v(\cdot))$ in problem $(\widehat{P}_1^V)$ with $x_0 \ge \varkappa$ for $\varkappa'\delta \le \rho$ does not depend on V for $V > \rho$ and is attained on the same optimal admissible process. The optimal value of this functional in problem $(\widehat{P}_1)$ is the limit of the optimal values of this functional in problems $(\widehat{P}_1^V)$ as $V \to \infty$,[2] and so it is attained on the same optimal admissible process.

It remains to consider case (c), when $\varkappa'\delta > \rho$. We have $\dot{x}(t) < 0$ in Γ_0 (see (8.8)) and $\dot{x}(t) > 0$ in Γ_1 for $V > \varkappa'\delta$ (see (7.1) for $v(t) = V$). Moreover, it follows from (8.6) that the trajectories cross Γ_{01} from Γ_0 to Γ_1 for $x > \underline{x}$ and from Γ_1 to Γ_0 for $x < \underline{x}$ (see Fig. 3). So for $x_0 \ge \underline{x}$ there is only one trajectory (of the Hamiltonian system of the PMP for problem $(\widehat{P}_1^V)$) with $x(0) = x_0$ that completely lies in Γ. This trajectory corresponds to a unique optimal solution in problem $(\widehat{P}_1^V)$, does not depend on V for $V > \varkappa'\delta$ and is the same as described above for problem $(\widehat{P}_1)$. Again, we conclude that this trajectory corresponds to an optimal admissible process in problem $(\widehat{P}_1)$. □

Remark 1. Note that for $x_0 < \varkappa$ in cases (a) and (b) and for $x_0 < \underline{x}$ in case (c), problem $(\widehat{P}_1^V)$ also has an optimal solution. The corresponding trajectory starts in Γ_1, reaches B_{01} or B in finite time, respectively, and we have $v_*(t) = V$ until the trajectory reaches B_{01} or B. In the limit $V \to \infty$, this yields an instantaneous jump to $x = \varkappa$ or $x = \underline{x}$, respectively, followed by the optimal solution in $\overline{\Gamma}_0$ described above (compare Theorem 6 below).

Now we are ready to formulate a final result.

[2] This is easy to check for the functional $\widetilde{J}_1(x(\cdot), u(\cdot), v(\cdot))$ (see (4.11)) due to estimate (6.2). Hence this is also valid for the functional $\widehat{J}_1(x(\cdot), u(\cdot), v(\cdot))$.

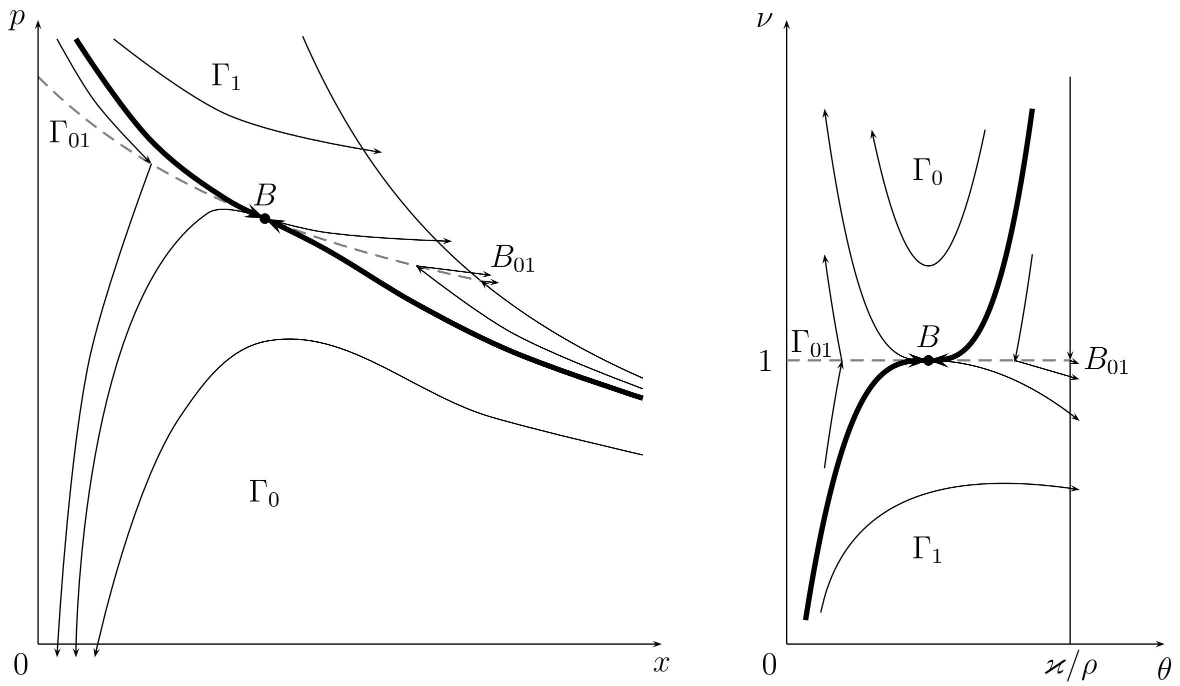

Figure 3: Trajectories of the Hamiltonian system of the PMP for problem $(\widehat{P}_1^V)$ in Γ for $(1-\varkappa)\delta > \rho$ (case (c)) in the coordinates (x,p) (left) and (θ,ν) (right). In $\overline{\Gamma}_0$ these trajectories coincide with those for problem $(\widehat{P}_1)$. Optimal trajectories are shown by thick lines, and Γ_{01} is shown by dashed lines

Theorem 5. *Let $\gamma = 1$ and $0 < \varkappa < 1$.*

(a) *If $\varkappa + \rho/\delta > 1$, then for $x_0 \geq \varkappa$ (i.e., $K_0^{1-\varkappa} \geq \varkappa S_0$) the unique optimal process in problem $(\widehat{P}_1)$ (or $(\widetilde{P}_1)$) has the form $x_*(t) = x_0 e^{(\rho-(1-\varkappa)\delta)t}$, $u_*(t) \equiv 0$, $v_*(t) \equiv \rho$ for $t \geq 0$. The corresponding optimal capital stock $K_*(\cdot)$ and resource stock $S_*(\cdot)$ in problem (P_1) are $K_*(t) = K_0 e^{-\delta t}$ and $S_*(t) = S_0 e^{-\rho t}$, $t \geq 0$, with $R_*(t) = \rho S_*(t) = \rho S_0 e^{-\rho t}$, $t \geq 0$. For $x_0 < \varkappa$ (i.e., $K_0^{1-\varkappa} < \varkappa S_0$) problem $(\widehat{P}_1)$ (as well as $(\widetilde{P}_1)$) has no optimal solutions in the class of locally integrable controls (in the class of distributions an optimal process starts with an instantaneous jump to the point $x = \varkappa$).*

(b) *If $\varkappa + \rho/\delta = 1$, then exactly the same conclusions as in (a) hold true, with the only difference that in this case $x_*(t) \equiv x_0$ for $t \geq 0$ if $x_0 \geq \varkappa$.*

(c) *If $\varkappa + \rho/\delta < 1$, then for any initial state $x_0 \geq \underline{x} := \varkappa(1-\varkappa)\rho/((1-\varkappa)\delta - \varkappa\rho)$ in problem $(\widehat{P}_1)$ (or $(\widetilde{P}_1)$), there is only one trajectory of the Hamiltonian system of the PMP with $x(0) = x_0$ that satisfies the necessary optimality conditions formulated in Theorem 2. This trajectory reaches the point $B = (\underline{x}, \underline{p})$ (8.7) in finite time and then stays at B. It corresponds to a unique optimal solution in problem $(\widehat{P}_1)$ (or $(\widetilde{P}_1)$). The x-coordinate varies monotonically along this trajectory, and so the p-coordinate can be expressed as a well-defined function of x: $p = p(x)$, $x > 0$. The optimal controls can then be represented in the form of optimal synthesis, $u_* = u_*(x)$ and $v_* = v_*(x)$, via formula (8.2) for $x > \underline{x}$ and formula (8.3) and the formula for $\widehat{v}$ given right after (8.6) for $x = \underline{x}$, with $p = p(x)$. In particular, we have $u_*(x) = 0$ for $x > \underline{x}$, $u_*(\underline{x}) = \varkappa((1-\varkappa)\delta - \rho)/((1-\varkappa)\delta - \varkappa\rho)$ and $v_*(\underline{x}) = \rho$.*

The qualitative behavior of the optimal trajectory of the Hamiltonian system of the PMP in case (c) *is shown in Fig.* 3.

For $x_0 < \underline{x}$ (*i.e.*, $K_0^{1-\varkappa} < \underline{x}S_0$) *problem* $(\widehat{P}_1)$ (*as well as* $(\widetilde{P}_1)$) *has no optimal solutions in the class of locally integrable controls* (*in the class of distributions an optimal process starts with an instantaneous jump to the point* $x = \underline{x}$).

8.2 The knife-edge case ($\gamma = 1$) with strong scale effects in capital accumulation ($\varkappa = 1$)

Again, note that all arguments presented in Section 7.2 up to and including the paragraph containing formula (7.25) remain valid for $\gamma = 1$. However, the problem of maximizing the first integral in (7.24) for $\gamma = 1$ under the constraint $\int_0^T R(t)\,dt = S_0 - S(T)$ has no solution in the class of locally integrable functions if $S(T) < S_0 - \rho T$. Indeed, due to the decreasing discount factor $e^{-\rho t}$, it is clear that, in an optimal regime, $R(\cdot)$ must be maximally "shifted" to the point $t = 0$, i.e., in the sense of distributions the solution is given by

$$R_*(t) = (S_0 - S(T) - \rho T)\delta_0(t) + \rho, \qquad 0 < t \le T,$$

where $\delta_0(\cdot)$ is the Dirac delta function. The corresponding value of the objective functional is obviously unattainable in the class of locally integrable functions.

Let us now determine the optimal values of T and $S(T)$. Denote $S_0 - S(T) - \rho T$ by X. Then

$$\begin{aligned}\widehat{J}_1(u_*(\cdot), R_*(\cdot)) &= \frac{X}{\rho} + \frac{\ln\rho}{\rho}\big[1 - e^{-\rho T}\big] + e^{-\rho T}\left[\frac{\ln S(T)}{\rho} + \frac{\ln\rho - 1}{\rho}\right] \\ &= \frac{1}{\rho}\big[X + \ln\rho - e^{-\rho T} + e^{-\rho T}\ln(S_0 - \rho T - X)\big], \end{aligned} \tag{8.9}$$

where $X \ge 0$ and $0 < S_0 - \rho T - X \le 1$. The derivative of this expression with respect to T is

$$e^{-\rho T}\left[1 - \ln(S_0 - \rho T - X) - \frac{1}{S_0 - \rho T - X}\right] \le 0;$$

hence the optimal $T_* = \max\{(S_0 - X - 1)/\rho, 0\}$. The derivative of the expression (8.9) with respect to X at $T = T_*$ is

$$\frac{1}{\rho}\left[1 - \frac{e^{-\rho T_*}}{S_0 - \rho T_* - X}\right];$$

hence the optimal $X_* = \max\{S_0 - \rho T_* - e^{-\rho T_*}, 0\}$.

If $T_* > 0$, then $\rho T_* = S_0 - X_* - 1 > 0$, so $0 \le X_* = S_0 - \rho T_* - 1 < S_0 - \rho T_* - e^{-\rho T_*} = X_*$, which is impossible. Therefore, $T_* = 0$ and $X_* = \max\{S_0 - 1, 0\}$. Hence, we arrive at the following conclusion.

Theorem 6. *For* $\gamma = \varkappa = 1$ *the optimal control* $R_*(\cdot)$ (*in the class of distributions*) *in problem* (P_1) *is*

$$R_*(t) = \max\{S_0 - 1, 0\}\delta_0(t) + \rho\min\{S_0, 1\}e^{-\rho t}, \qquad t \ge 0,$$

and $u_*(0) = 1$, $u_*(t) = 0$ *for* $t > 0$. *The corresponding optimal resource stock* $S_*(\cdot)$ *and capital stock* $K_*(\cdot)$ *are* $S_*(t) = \min\{S_0, 1\}e^{-\rho t}$ *and* $K_*(t) = K_0 e^{\max\{S_0-1,0\}-\delta t}$ *for* $t > 0$, *and the corresponding* (*maximum*) *value of the objective functional* $\widehat{J}_1(u_*(\cdot), R_*(\cdot))$ *is*

$$\frac{1}{\rho}\big[\max\{S_0 - 1, 0\} + \ln\rho - 1 + \ln\min\{S_0, 1\}\big],$$

which is unattainable in the class of locally integrable controls $R(\cdot)$ *if* $S_0 > 1$.

In the form of optimal synthesis, the optimal strategy can informally be formulated as utilizing the resource amount of $S_0 - 1$ *immediately* (*and investing it completely in increasing the capital stock*) *if* $S_0 > 1$, *and then extracting the resource according to the Hotelling rule* ($R_*(t) = \rho S_*(t)$).

Proof of Theorem 1(ii) *in the case of* $\gamma = 1$ *and* $\varkappa = 1$. The required assertion is contained in Theorem 6. □

This completes the proof of Theorem 1.

9 A sensitivity analysis of optimal policies

The above theorems provide solutions to the depreciation-augmented DHSS model for all combinations of the model parameters. In all cases we described the qualitative and asymptotic (as $t \to \infty$) behavior of optimal processes, and in some simple cases we were able to explicitly obtain the optimal trajectory. Let us identify plausible scenarios implied in the analysis of the Hamiltonian system and perform a sensitivity analysis of the model solutions to model parameters.

The model has six parameters: two output elasticities γ and $\varkappa$, the discount rate ρ, the depreciation rate δ and the initial values of the stocks K_0 and S_0. Can we narrow the magnitudes of the model parameters on a priori grounds? Perhaps not the magnitudes of the elasticities, whose sum is our main discriminatory variable, nor those of the initial stocks, as they are highly uncertain. Yet we have compelling reasons to assume $\rho < \delta$. The rate of capital depreciation reflects the decrease in the productivity of a capital asset over time. The productivity of tangible assets deteriorates with wear and tear, and most assets suffer from technological obsolescence. The U.S. Bureau of Economic Analysis (BEA) provides estimates of capital depreciation rates by asset types, most of which are based on the market prices of aged capital assets. The average estimate for equipment, structures and intellectual property capital increased from 0.06 (27y) in 1947 to 0.1 (21y) in 2014, where the figures in parentheses show the approximate service life-times in years. Thus, during the last seven decades, the economic life-time of an average asset has decreased by roughly six years. This decrease owes to the emergence of new assets related to information and communication technologies, and to the rising importance of intangible goods such as intellectual property rights, both types being relatively short-lived. The BEA estimates for 2014 put the depreciation rate for computers at 0.33 (5y) and at 0.3 (6y) for intellectual property assets. The depreciation rate for intellectual property has decreased from 0.13 (15y) in 1947 to 0.3 (6y) in 2014. If the recent pace of technological progress continues, the Schumpeterian process of creative destruction is likely to further shorten economic life-times, thereby also increasing the aggregate depreciation rate. The inclusion of R&D capital will likely further increase the aggregate depreciation rate, as estimates typically place the depreciation rate for R&D capital between roughly 0.2 and 0.4 (Li [38]). Together, these facts suggest that depreciation rates tend to be higher in richer countries, and that the global rate of depreciation should increase as a consequence of economic growth and the diffusion of capital goods and knowledge. On the contrary, the aggregate depreciation rate may decrease if we include human capital, measured by the educational attainment of labor. Human capital depreciates due to death or disability, and because skills can become obsolete or lost in a prolonged period of unemployment. The estimated rates of depreciation of human capital range from 0.01 to 0.015 (Groot [27], Arrazola and Hevia [3]).

The above evidence makes an aggregate depreciation rate of 0.1 plausible. The notion of discounting is contentious, as it implies a higher valuation of our generation's consumption over that of future generations. A survey by Harrison [31] shows that constant discount rates typically used in cost-benefit analysis range between 0.01 and 0.08. For example, the influential review by Stern [48] and the reply by Nordhaus [41] used discount rates of 0.014 and 0.055, respectively. Discount rates are thus typically lower than depreciation rates, a relationship that we will maintain in the discussion below.

Consider the main case $0 < \varkappa < 1$, $0 < \gamma < 1$, which admits decreasing, constant and increasing returns to scale, and $\delta > 0$. The maximum increasing returns that fall under this are arbitrarily close to 2. The existence theorem in Section 6 shows that an optimal solution exists for all parameter values. Theorem 3 shows that the asymptotic optimal investment and depletion policies depend on the value of $\varkappa + (\rho/\delta)\gamma$. Fixing an inequality between the discount rate and the depreciation rate allows us to anchor the discussion of different solutions in the main case to the magnitude of returns to scale. So we assume below that $\rho < \delta$.

The following two cases are possible:

(i) $\varkappa + (\rho/\delta)\gamma < 1$: decreasing, constant or weakly increasing returns, in which case the optimal investment-to-output ratio tends to a fixed number and the optimal depletion rate tends to the discount rate;

(ii) $\varkappa + (\rho/\delta)\gamma \geq 1$: strongly increasing returns, which lead to a pure depletion strategy with asymptotically zero optimal investment and with an optimal depletion rate asymptotically equal to the discount rate.

Case (i) is more relevant in view of the existing empirical evidence for nearly-constant or slightly increasing returns at an aggregate level.

9.1 Decreasing, constant or weakly increasing returns to scale

The optimal solution for $\varkappa + (\rho/\delta)\gamma < 1$ is given by the last case in Theorem 3. In this case, the optimal policies converge to a constant share of output invested in capital and a constant rate of depletion of the natural resource, as $t \to \infty$. The asymptotic optimal rate of depletion equals the discount rate, $v_*(t) \to \rho$, which is the well-known Hotelling rule. The asymptotic optimal investment ratio is given by

$$u_*(t) \to u_\star(\varkappa, \gamma, \rho, \delta) = 1 - \frac{(1-\varkappa)((1-\gamma)\rho + (1-\varkappa)\delta)}{(1-\varkappa-\gamma)\rho + (1-\varkappa)\delta}$$

$$= \varkappa - \frac{\varkappa(1-\varkappa)\rho}{(1-\varkappa-\gamma)\rho + (1-\varkappa)\delta} = \frac{\varkappa((1-\varkappa)\delta - \gamma\rho)}{(1-\varkappa-\gamma)\rho + (1-\varkappa)\delta}. \quad (9.1)$$

The partial derivatives with respect to δ, ρ and γ have the following signs:

$$\frac{\partial u_\star(\varkappa, \gamma, \rho, \delta)}{\partial \delta} = \frac{\varkappa(1-\varkappa)^2\rho}{((1-\varkappa-\gamma)\rho + (1-\varkappa)\delta)^2} > 0,$$

$$\frac{\partial u_\star(\varkappa, \gamma, \rho, \delta)}{\partial \rho} = \frac{-\varkappa(1-\varkappa)^2\delta}{((1-\varkappa-\gamma)\rho + (1-\varkappa)\delta)^2} < 0, \quad (9.2)$$

$$\frac{\partial u_\star(\varkappa, \gamma, \rho, \delta)}{\partial \gamma} = \frac{-\varkappa(1-\varkappa)\rho^2}{((1-\varkappa-\gamma)\rho + (1-\varkappa)\delta)^2} < 0.$$

As expected, the asymptotic investment share is an increasing function of the depreciation rate and a decreasing function of the discount rate. Faster decay of capital and higher valuation of future output (lower discount rate) require higher investment effort. The partial derivative with respect to the output elasticity of resource γ is negative, which implies that a greater sensitivity of output to changes in the natural input corresponds to a lower investment rate. This is consistent with the higher investment rates typically observed in developed economies that employ higher stocks of produced capital. The relationship between the asymptotic optimal investment ratio and the output elasticity of produced capital $\varkappa$ is ambiguous. The corresponding partial derivative takes both positive and negative values in the relevant parameter range. In particular, it follows from the last expression in (9.1) that the partial derivative is negative for $\varkappa + (\rho/\delta)\gamma$ close to 1 (i.e., for $(1-\varkappa)\delta$ close to $\gamma\rho$) and is positive for $\varkappa$ close to zero.

Note that the above parametrization includes constant returns to scale as a special case ($\varkappa + \gamma = 1$). Then (see (9.1))

$$u_\star(\varkappa, \gamma, \rho, \delta) = u_\star(\varkappa, 1-\varkappa, \rho, \delta) = \frac{\varkappa(\delta - \rho)}{\delta}.$$

Take a depreciation rate $\delta = 0.1$ and a discount rate $\rho = 0.05$, which is roughly the average discount rate used in the studies surveyed by Harrison [31]. Then $u_\star(\varkappa, \gamma, \rho, \delta) = \varkappa/2$. For example, the asymptotic optimal investment ratio would equal 0.35 in an economy with high elasticity with respect to produced capital ($\varkappa = 0.7$, $\gamma = 0.3$) and equal 0.15 when the elasticity with respect to the resource is high ($\varkappa = 0.3$, $\gamma = 0.7$).

9.2 Strongly increasing returns to scale

Assuming $\rho < \delta$ and $\varkappa + (\rho/\delta)\gamma \geq 1$ leads to the case of strongly increasing returns to scale, with the corresponding solutions contained in the first two cases in Theorem 3. The optimal policies entail a pure depletion strategy with zero investment in produced capital. However, whether a zero investment strategy is followed from the outset, from some instant of time or asymptotically will depend on the relative sizes of the two stocks. It is convenient to interpret the case in terms of the hypothetical current capital coefficient $K(t)/Y(K(t), S(t))$, which is the current value of the state variable $x(\cdot)$ at time $t \geq 0$ in the model. The actual current capital coefficient is given by $K(t)/Y(K(t), R(t))$.

For $\varkappa + (\rho/\delta)\gamma > 1$, a pure depletion strategy becomes optimal as soon as

$$\frac{K_*(t)}{Y(K_*(t), S_*(t))} \geq \frac{\varkappa}{\rho^{1-\gamma}}. \tag{9.3}$$

If this inequality is not valid at $t = 0$, then it is always achieved at some finite moment in an optimal solution. Due to the smallness of the discount rate ρ, this scenario corresponds either to a currently mature economy with a sizable capital stock ($K_*(t)$ is large) or to an economy nearing the complete exhaustion of the natural resource ($S_*(t)$ is small), or to an economy with both properties.

The knife-edge case $\varkappa + (\rho/\delta)\gamma = 1$ also involves a pure depletion strategy that can be optimal or asymptotically optimal, but now this depends on the initial sizes rather than the current sizes of the two stocks. If inequality (9.3) is not valid at $t = 0$, then it will not be achieved at a finite moment, but only asymptotically.

9.3 Strong scale effects in capital accumulation

To conclude, we consider special cases in which either $\varkappa = 1$ or $\gamma = 1$. In the latter case, increasing returns emanate from the use of exhaustible resources, whereas in the former case it is the produced capital that is responsible for the (strongly) increasing returns. In the case $\varkappa = 1$, we say that the accumulation of produced capital is subject to strong scale effects. These knife-edge cases lead to markedly different optimal investment and depletion policies that depend on the model parameters as well as the initial capital and resource stocks, and may lead to the nonexistence of a solution.

Theorem 4 summarizes the model solutions in the case of strong scale effects in capital accumulation, when $\varkappa = 1$ and $\gamma < 1$. The optimal depletion policy expressed in terms of the depletion rate of the natural stock can be written as

$$v_*(S_*(t), \gamma, \rho) = \max\left\{\rho,\ \frac{\rho}{1-\gamma} - \frac{\gamma\rho^{1/\gamma}}{(1-\gamma)S_*(t)}\right\}. \tag{9.4}$$

Due to the nature of the variables involved, we can restrict ourselves to the more plausible situation when $S_*(t) > \rho^{(1-\gamma)/\gamma}$. Then the second argument of the maximum in (9.4) prevails.

The optimal extraction rate is an increasing function of the current resource stock $S_*(t)$ (provided that $S_*(t) \geq \rho^{(1-\gamma)/\gamma}$). The greater the stock, the greedier the depletion policy. The optimal rate of depletion is an increasing function of the discount rate ρ, since

$$\frac{\partial v_*(S_*(t), \gamma, \rho)}{\partial \rho} = \frac{S_*(t) - \rho^{(1-\gamma)/\gamma}}{S_*(t)(1-\gamma)} > 0, \qquad t \geq 0.$$

Interestingly, the optimal depletion rate becomes the Hotelling rate $v_*(t) = \rho$ just before exhaustion, namely, when $S_*(t) \leq \rho^{(1-\gamma)/\gamma}$.

The optimal investment depends on the optimal quantity extracted:

$$u_*(S_*(t), \gamma, \rho) = 1 - \left(\frac{1-\gamma}{\rho^{-(1-\gamma)/\gamma} S_*(t) - \gamma} \right)^{\gamma} \qquad \text{for} \quad S_*(t) > \rho^{(1-\gamma)/\gamma}$$

(and $u_*(t) = 0$ for $S_*(t) \leq \rho^{(1-\gamma)/\gamma}$); i.e., the optimal share of final output invested in produced capital becomes zero just before exhaustion.

It is clear that the dynamic investment policy is an increasing function of the current resource stock $S_*(t)$ and a decreasing function of the discount rate ρ. Thus, the higher the current resource endowment, the higher the investment effort.

9.4 Strong scale effects in resource use

The existence theorem in Section 6 shows that a pair of welfare-maximizing investment and extraction policies may not exist if the output elasticity of the resource γ equals 1. Optimal polices will not exist when an initial stock of produced capital K_0 is small relative to the initial resource stock S_0, or equivalently when the hypothetical initial capital coefficient is small:

$$\frac{K_0}{Y(K_0, S_0)} < \mu := \begin{cases} \dfrac{\varkappa(1-\varkappa)\rho}{(1-\varkappa)\delta - \varkappa\rho}, & \varkappa < 1 - \dfrac{\rho}{\delta}, \\ \varkappa, & \varkappa \geq 1 - \dfrac{\rho}{\delta}. \end{cases}$$

This implies, curiously enough, that it would have been impossible to formulate a welfare-maximizing policy in the early history of humanity, when produced capital was scarce and resources were abundant. This only becomes possible when the produced stock has reached a certain level. In the model, this implies an initial jump to a level of produced capital that allows formulating such a policy thereafter.

For the case $\varkappa < 1$ and $\gamma = 1$, Theorem 5 shows that the optimal policies depend on the relationship between the output elasticity of capital $\varkappa$ and the ratio of the discount rate to the depreciation rate, ρ/δ, which can be assumed to be smaller than 1.

Assume that the initial lower bound on the capital coefficient is greater than μ, so that the initial stock of produced capital is not too small and an optimal solution exists.

Then, for $\varkappa \geq 1 - \rho/\delta$, the optimal solution is a pure depletion policy from the outset, with the optimal depletion rate following the well-known Hotelling rule, according to which the extraction rate of an exhaustible resource should equal the discount rate.

In the case $\varkappa < 1 - \rho/\delta$, the optimal investment and depletion policies become constant as soon as the capital coefficient reaches the value

$$\mu = \underline{x} = \frac{\varkappa(1-\varkappa)\rho}{(1-\varkappa)\delta - \varkappa\rho}.$$

Note that this happens in finite time, and then the capital coefficient remains unchanged. The optimal share of final output invested in capital after that instant of time is given by

$$u_*(\underline{x}) := u_\star(\varkappa, \delta, \rho) = \frac{\varkappa((1-\varkappa)\delta - \rho)}{(1-\varkappa)\delta - \varkappa\rho} = \varkappa - \frac{\varkappa(1-\varkappa)\rho}{(1-\varkappa)\delta - \varkappa\rho} = 1 - \frac{(1-\varkappa)^2\delta}{(1-\varkappa)\delta - \varkappa\rho}$$

(compare (9.1)). It is clear that

$$\frac{\partial u_\star(\varkappa,\delta,\rho)}{\partial\delta} > 0, \qquad \frac{\partial u_\star(\varkappa,\delta,\rho)}{\partial\rho} < 0$$

(see also (9.2) for $\gamma = 1$). Thus, the asymptotic investment share is an increasing function of the depreciation rate and a decreasing function of the discount rate. As we already observed in the analysis of the main case, faster decay of capital and higher valuation of future output induces higher investment. The partial derivative with respect to the output elasticity of capital $\varkappa$,

$$\frac{\partial u_\star(\varkappa,\delta,\rho)}{\partial\varkappa} = \frac{\delta(1-\varkappa)[\delta-\rho-\varkappa(\delta+\rho)]}{((1-\varkappa)\delta-\varkappa\rho)^2},$$

can be both negative and positive. If $\rho < \delta$, it is positive for $\varkappa < (\delta-\rho)/(\delta+\rho)$ and negative for $\varkappa > (\delta-\rho)/(\delta+\rho)$, so the investment share is maximal for $\varkappa = (\delta-\rho)/(\delta+\rho)$. For example, for $\rho = 0.05$ and $\delta = 0.1$ the investment share is maximal for the output elasticity of capital $\varkappa = 1/3$.

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